George M. Bergman
308 LeConte Hall

Spring 1995, Math 110, Section 1

**Final Examination**

15 May, 1995
8:10-11:00 AM

**1.** (a) (5 points) If V is a finite-dimensional vector space over a field $F$, and $\beta = \{x_1, \dots, x_m\}$ an ordered basis for V, define what is meant by the *coordinate vector* $[x]_\beta$ of an element $x \in V$ with respect to $\beta$. Also indicate, if it is not clear from your definition, why this coordinate vector is well-defined.

(b) (5 points) Suppose V and W are finite-dimensional vector spaces over a field $F$, with ordered bases $\beta = \{x_1, \dots, x_m\}$ and $\gamma = \{y_1, \dots, y_n\}$, and that $T: V \to W$ is a linear map. Define what is meant by $[T]_\beta^\gamma$, and give (without proof) a formula for the coordinate vector of the image under T of a vector $x \in V$ in terms of this matrix and the coordinate vector of $x$.

**2.** Suppose that S and T are two linear operators on a finite-dimensional vector space V which *commute*, i.e., satisfy $ST = TS$. To avoid confusion, for $\lambda \in F$ we will write $E_{T,\lambda}$ for $N(T - \lambda I)$ and $E_{S,\lambda}$ for $N(S - \lambda I)$.

(a) (5 points) Show that for every $\lambda \in F$, the subspace $E_{T,\lambda} \subseteq V$ is S-invariant.

(b) (10 points) Show that if T is diagonalizable, and if for each eigenvalue $\lambda$ of T, the operator $S_{E_{T,\lambda}}$ (the restriction of S to $E_{T,\lambda}$) is diagonalizable, then there exists an ordered basis $\beta$ of V such that $[S]_\beta$ and $[T]_\beta$ are both diagonal.

**3.** Let V be a finite-dimensional inner product space. For each $x \in V$, let us define $A(x) \in V^*$ by $A(x)(y) = \langle y, x \rangle$ $(y \in V)$. (You are *not* asked to verify that these maps are in fact members of $V^*$.)

(a) (12 points) Show that if V is a finite-dimensional *real* inner product space, the map $A: V \to V^*$ is linear and invertible.

(b) (3 points) Say briefly why the above result fails if V is a *complex* inner product space.

**4.** Let V be an inner product space, and $W_1$, $W_2$ nonzero subspaces such that $V = W_1 \oplus W_2$ (i.e., every element of V can be written uniquely as the sum of an element of $W_1$ and an element of $W_2$), and such that every element of $W_1$ is orthogonal to every element of $W_2$.

Let $T_1, T_2 : V \to V$ be the linear maps defined by $T_1(x_1 + x_2) = x_1$, $T_2(x_1 + x_2) =$

$x_2$ for $x_1 \in \mathsf{W}_1$, $x_2 \in \mathsf{W}_2$ (the projection maps associated with this direct sum decomposition).

(a) (10 points) Show that for all $p, q \in F$, the operator $\mathsf{T} = p\mathsf{T}_1 + q\mathsf{T}_2$ has for adjoint the operator $\mathsf{T}' = \bar{p}\mathsf{T}_1 + \bar{q}\mathsf{T}_2$.

(b) (10 points) Find and prove necessary and sufficient conditions on $p, q \in F$ for the operator $\mathsf{T}$ defined in part (a) to be *orthogonal* (i.e., to satisfy $\langle x, y\rangle = \langle \mathsf{T}(x), \mathsf{T}(y)\rangle$ for all $x, y \in \mathsf{V}$).

5. Let $\mathsf{T}$ be a linear operator on a finite-dimensional vector space $\mathsf{V}$, let $\lambda$ be an eigenvalue of $\mathsf{V}$, and suppose that the generalized eigenspace $\mathsf{K}_\lambda$ has a Jordan canonical basis $\{x_1, \ldots, x_9\}$, with dot diagram

| | | | |
|---|---|---|---|
| $x_1$ · | $x_5$ · | $x_8$ · | $x_9$ · |
| $x_2$ · | $x_6$ · | | |
| $x_3$ · | $x_7$ · | | |
| $x_4$ · | | | |

(a) (7 points) Write out formulas showing the action of $\mathsf{T}$ on $\mathsf{K}_\lambda$ in terms of this basis. (If you are unsure, it may help to start by writing out formulas for the action of $\mathsf{T} - \lambda\mathsf{I}$.)

(b) (7 points) Give the matrix for $\mathsf{T}_{\mathsf{K}_\lambda}$ (the restriction of $\mathsf{T}$ to $\mathsf{K}_\lambda$) in terms of the basis $\{x_1, \ldots, x_9\}$. (You do not have to write out the 0's, as long as you make it clear what the other matrix entries are, and where they go.)

(c) (8 points) Give the dot diagram of a Jordan canonical basis for the restriction of the operator $\mathsf{T}$ to the subspace $\mathsf{N}((\mathsf{T} - \lambda\mathsf{I})^2) \subseteq \mathsf{K}_\lambda$, naming the basis elements.

(d) (8 points) Give the dot diagram of a Jordan canonical basis for the restriction of $\mathsf{T}$ to $(\mathsf{T} - \lambda\mathsf{I})^2(\mathsf{K}_\lambda)$, i.e., the subspace $\{(\mathsf{T} - \lambda\mathsf{I})^2(x) \mid x \in \mathsf{K}_\lambda\}$, again labeling the basis elements.

6. (10 points) Let $\mathsf{T}$ be a linear operator on a finite-dimensional vector space $\mathsf{V}$, and let $\mathsf{W}$ be a $\mathsf{T}$-invariant subspace of $\mathsf{V}$. Show that the minimal polynomial of $\mathsf{T}_\mathsf{W}$ (the restriction of $\mathsf{T}$ to $\mathsf{W}$) divides the minimal polynomial of $\mathsf{T}$.