Zworski  Spr 2004

MATH 1B FINAL EXAM

Please try to answer each question on the page on which it is presented, including the back of the page. Generous partial credit will be given.

Good luck!

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Problem 1. (12 points) Evaluate the following integrals

a) \[ \int e^x \cos^2(e^x - 1) \, dx \]

b) \[ \int \frac{\cos x}{\sin^2 x + 3 \sin x + 2} \, dx \]

c) \[ \int_0^1 \frac{\sqrt[3]{x^3} + \sqrt{x}}{\sqrt{x}} \, dx \]
Problem 2. (12 points) Solve the following differential equations

a) \( y'' - y = e^x, \quad y(0) = 0, \quad y'(0) = 1 \)

b) \( y' = \frac{1}{y(1 + x^2)}, \quad y(0) = 2 \)

c) \( y' + x^2 y = x^5, \quad y(0) = 0. \)
Problem 3. (12 points)
a) Define a direction field associated to a first order differential equation.
b) Match four direction fields in the figure files A-F to four of the following differential equations:

1) \( y' = (1 - y^2) \sin x \)
2) \( y' = \sin x \)
3) \( y' = x^3 - y^3 \)
4) \( y' = \sin(x + y) \)
5) \( y' = x^2 + y^2 \)
6) \( y' = x^2 - y^3 \)

Please note that in the figures the coordinates \((x, y)\) are not on the axes but on the boundaries of the figure and that the axes are scaled differently. You will lose a point for each wrong match.
Problem 4. (12 points) This is a multiple choice problem. You do not need to show any work but you will lose one point for each wrong answer.

1) The series
\[ \sum_{n=0}^{\infty} 2^{-n} \sin (3^n) \]
a) diverges b) converges by the alternating series test c) converges by the root test
d) converges by the comparison test e) converges by the integral test?

2) The radius of convergence of
\[ \sum_{n=1}^{\infty} \frac{x^n}{2^n n^2} \]
is a) 0 b) 1 c) 2 d) \( \infty \) e) none of the above?

3) The radius of convergence of
\[ \sum_{n=1}^{\infty} \frac{x^n}{2^n n^2} \]
is a) 0 b) 1 c) 2 d) \( \infty \) e) none of the above?

4) The initial value problem
\[ y'' - x^4 y' = 0, \quad y(0) = 0, \quad y'(0) = 1, \]
is solved by the following power series:

\[ a) \sum_{n=1}^{\infty} \frac{x^{4n-1}}{(4n-1)n!} \]
\[ b) \sum_{n=1}^{\infty} \frac{x^{4n+1}}{(4n+1)n!} \]
\[ c) \sum_{n=1}^{\infty} \frac{x^{4n-3}}{(4n-3)(n-1)!} \]
\[ d) \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n+1)n!} \]
or e) none of the above.
Problem 5. (14 points)
a) Consider the following non-homogeneous equation:
\[ y'' + by' + cy = G(x) . \]

Explain and derive the equations appearing in the method of variation of parameters:
\[ u'_1y_1 + u'_2y_2 = 0 \]
\[ u'_1y'_1 + u'_2y'_2 = G(x) . \]

If you cannot derive the equations, explain clearly what they mean!
b) Find the general solution of the equation
\[ y'' + y = e^x . \]
c) Find the general solution of the equation
\[ y'' - y = \frac{1}{1 + e^x} . \]
Problem 6. (12 points)
a) Define an alternating series and state the conditions under which it converges.
b) Which of the following series converge (provide clear explanations):
\[ 1) \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2} \quad 2) \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n} \quad 3) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \quad 4) \sum_{n=1}^{\infty} \frac{\sin(n\pi)}{n} \]
c) State the error estimate for the alternating series and find out how many terms in the convergent series in part b) are needed to obtain an approximation with an error < 0.0001.
Problem 7. (12 points)
a) Suppose that \( f(x) = \sum_{n=1}^{\infty} a_n x^n \), where the series converges for all \( x \). What is the formula for the coefficients \( a_n \)?
b) Suppose that \( T_n(x) \) is the \( n \)-th degree Taylor polynomial of \( f(x) \) at \( x = 0 \). Give an estimate \( |f(x) - T_n(x)| \) for \( |x| \leq R \).
c) Find the \( n \)th degree Taylor polynomial of \( \cosh x \) (recall that \( \cosh x = (e^x + e^{-x})/2 \)).
d) Approximate \( \cosh(0.1) \) with an error less than 0.0001.
Problem 8. (14 points)

a) Let \( f(x) \) be a function defined for \( a \leq x \leq b \). What is the arclength of the graph of \( f \)?

b) State Simpson's rule with \( a = 0 \), \( b = 1 \), and \( n = 2 \). That is, approximate the integral \( \int_0^1 g(x) \, dx \) using three values of \( g \): \( g(0) \), \( g(1/2) \), and \( g(1) \). Some credit will be given for the correct error bound.

c) Use Simpson's rule of part b) (or for partial credit any rule for approximate integration) to approximate the length of the graph of \( \cos(\pi x)/\pi \) between \( x = 0 \) and \( x = 1 \). Compare your answer to the exact length, 1.2160\( \cdots \).