MATH 185 FINAL

Do not write your answers on this sheet. Instead please write your name, your student ID, and all your answers in your blue books. Total: 100 pts., 2 hours and 50 minutes.

(1) (4 pts. each) For each of (a)-(e) below: If the proposition is true, write TRUE and explain why it is true. If the proposition is false, write FALSE and give a counterexample. (Please do not use the abbreviations T and F, since in handwriting they are sometimes indistinguishable.)

(a) If $f: \mathbb{C} \to \mathbb{C}$ is a function such that the functions $\text{Re} f(x + iy)$ and $\text{Im} f(x + iy)$ are differentiable (as functions $\mathbb{R}^2 \to \mathbb{R}$) at every point $(x, y) \in \mathbb{R}^2$, then $f$ is an entire function.

(b) If $\ell(z)$ is a branch of $\log z$ on a domain $G$, and $z_1, z_2 \in G$ satisfy $z_1/z_2 \in G$, then $\ell(z_1/z_2) = \ell(z_1) - \ell(z_2)$.

(c) If $\gamma$ is a closed curve contained in the set $\mathbb{C}^*$ of nonzero complex numbers, then

$$\int_{\gamma} \frac{1}{z^2} \, dz = 0.$$ 

(d) If a Laurent series centered at 0 converges at $4i$, then it converges also at $-1$.

(e) If $R > 0$, the domain $G_R := \{z \in \mathbb{C} : |z| > R\}$ is simply connected.

(2) (6 pts.) Let $a$ and $b$ be distinct complex numbers. Find, in terms of $a$ and $b$, all complex numbers $c$ such that $a, b, c$ are the vertices of a triangle with a $30^\circ$ angle at $a$, a $60^\circ$ angle at $b$, and a $90^\circ$ angle at $c$.

(3) (5 pts.) Sketch the set of $z \in \mathbb{C}$ such that $\text{Re}(e^z) < 0$.

(4) (8 pts.) Does the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converge uniformly on $\mathbb{C}$? (Justify your answer.)

(5) (8 pts.) Let $f(z)$ and $g(z)$ be functions that are holomorphic on a neighborhood of 0, and assume that $g(z)$ has a zero of order 2 at $z = 0$. Let $a_0 = f(0)$, $a_1 = f'(0)$, $b_2 = g''(0)$, $b_3 = g'''(0)$. Find a formula for the residue of $f(z)/g(z)$ at $z = 0$ in terms of $a_0, a_1, b_2, b_3$.

(6) (8 pts.) Let $G = \{z \in \mathbb{C} : |z - 2i| > 1\}$. Let $\gamma_0$ be the straight line path from 4 to $-4$, and let $\gamma_1$ be the path $\gamma_1(t) = 4e^{it}$ for $t \in [0, \pi]$. Are $\gamma_0$ and $\gamma_1$ (fixed-point) homotopic in $G$? Use complex integration to prove your answer.

(more problems on back)
(7) (12 pts. each) Evaluate the following definite integrals:

(a) \[ \int_{0}^{2\pi} \frac{1}{5 + 3\sin \theta} d\theta \]

(b) \[ \int_{-\infty}^{\infty} \frac{1}{(z^2 + a^2)(z^2 + b^2)} \, dz \text{ where } 0 < a < b. \]

(8) (10 pts.) Let \( f(z) \) be a function that is holomorphic in the region where \( 0 < |z| < 1 \). Assume that \( f(1/n) = 0 \) for all integers \( n \geq 2 \), but \( f(z) \) is not identically zero. Prove that \( f(z) \) has an essential singularity at \( z = 0 \).

(9) (8 pts.) Let \( n \geq 1 \), and let \( a_0, a_1, \ldots, a_n \) be complex numbers such that \( a_n \neq 0 \). For \( \theta \in \mathbb{R} \), define

\[ f(\theta) = a_0 + a_1 e^{i\theta} + a_2 e^{2i\theta} + \cdots + a_n e^{ni\theta}. \]

Prove that there exists \( \theta \in \mathbb{R} \) such that \( |f(\theta)| > |a_0| \).

(10) (3 pts.) Let \( \gamma \) be the closed curve illustrated below, and let \( c \) be the complex number marked by the \( * \). What is \( \text{ind}_\gamma(c) \)?

This is the end! At this point, you may want to look over this sheet to make sure you have not omitted any problems. In particular, note that problem 1 has five parts, and problem 7 has two parts. Check that your answers make sense! Please take this sheet with you as you leave.