Math 113 Midterm #2, 4/10/03, 8:00 – 9:30 AM  M. Hutchings

NAME __________________________ Score __________

To receive full credit you must justify all answers except where otherwise stated. The point is to demonstrate that you understand the material. No books, notes, calculators, collaboration, or other aids are permitted. There are 5 questions, each with two parts worth 5 points each. Please write your answers on the exam, not in a blue book. You may use the backs of the pages if necessary. Good luck!

1. True or false (and of course justify):

   (a) If $R$ is an integral domain with quotient field $Q$ then the quotient field of $R[x]$ is isomorphic to $Q[x]$.

   (b) The group $\mathbb{Z}_4 \times \mathbb{Z}_{18}$ is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_{36}$.  

2. Let $G$ be a group. Consider the "diagonal"

$$H = \{(x, x) \mid x \in G\} \subset G \times G.$$  

$H$ is a subgroup of $G \times G$; you don't have to prove this.

(a) Show that $H$ is a normal subgroup of $G \times G$ if and only if $G$ is abelian.

(b) Assuming $G$ is abelian, show that $(G \times G)/H \simeq G$. (Hint: define a surjective homomorphism $\phi : G \times G \to G$ whose kernel is $H$.)
3. (a) Find all solutions to the equation \( x^2 - 1 = 0 \) in \( \mathbb{Z}_{35} \).

(b) Show that if \( p > 2 \) is prime then either \( 2^{(p-1)/2} + 1 \) or \( 2^{(p-1)/2} - 1 \) is a multiple of \( p \). (Hint: consider the order of 2 in \( \mathbb{Z}_p^* \).)
4. (a) Find the quotient and the remainder when $x^3 + 8x^2 + 7x - 1$ is divided by $4x - 1$ in $\mathbb{Z}_{11}[x]$.

(b) Prove the "remainder theorem": if $F$ is a field, $p \in F[x]$, and $\alpha \in F$, then $p(\alpha)$ is the remainder when $p$ is divided by $x - \alpha$. (Here $p(\alpha)$ denotes the image of $p$ under the evaluation homomorphism $i_\alpha : F[x] \rightarrow F$.)
5. True or false (and of course justify):

(a) The quotient group \((\mathbb{Z} \times \mathbb{Z})/\langle(2, 4)\rangle\) is isomorphic to \(\mathbb{Z}\).

(b) There exists a nonzero homomorphism from the group \(\mathbb{Z}_{33}\) to the group \(\mathbb{Z}_{20}\).