Math 128A Final 2003 May 21.  R. Borcherds

Please make sure that your name is on everything you hand in. You are allowed calculators and 1 page of notes. All questions have about the same number of marks.

1. Determine the free cubic spline that approximates the data \( f(-1) = 1, \ f(0) = 0, \ f(1) = 1. \)

2. Use the modified Euler method

\[
  w_{i+1} = w_i + \left( \frac{h}{2} \right) (f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i)))
\]

to approximate the solution to \( y' = 1 + (t - y)^2, \ 2 \leq t \leq 3, \ y(2) = 1 \) with \( h = \cdot 5. \)

3. Find constants \( a, b, c \) so that the formula

\[
  \int_0^2 f(x)dx = af(0) + bf(1) + cf(2)
\]

is exact whenever \( f \) is a polynomial of degree at most 2. (You should show how to derive these constants: just quoting them will not get much credit.)

4. Derive the formula \( w_{i+1} = w_i + h((3/2)f(t_i, w_i) - (1/2)f(t_{i-1}, w_{i-1})) \) for the Adams-Bashforth two step explicit method by using the Lagrange form of the interpolating polynomial.

5. Estimate \( y(1) \) where \( y' = -10y \) and \( y(0) = 1 \) using the forward Euler method \( w_{i+1} = w_i + hf(t_i, w_i) \) and the backward Euler method \( w_{i+1} = w_i + hf(t_{i+1}, w_{i+1}) \) with step size \( h = \cdot 5. \) Which method is better for this example?

6. For the following linear system

\[
  \begin{align*}
    x - ay &= 1 \\
    ax - y &= -1
  \end{align*}
\]
describe for which values of $a$ the system has an infinite number of solutions, no solutions, and exactly one solution, and find the solution when it is unique.

7. Write $A$ in the form $LDL^t$ where $A$ is

$$
\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{pmatrix},
$$

$L$ is lower triangular with 1’s on the diagonal, and $D$ is diagonal.

8. Find a permutation matrix $P$, a lower triangular matrix $L$ with 1’s on the diagonal, and an upper triangular matrix $U$ so that $PA = LU$ where $A$ is

$$
\begin{pmatrix}
1 & 2 & -1 \\
2 & 4 & 0 \\
0 & 1 & -1
\end{pmatrix}.
$$

9. Use the Runge-Kutta method of order 4 given by

$$
\begin{align*}
    k_1 &= hf(t_i, w_i) \\
    k_2 &= hf(t_i + h/2, w_i + k_1/2) \\
    k_3 &= hf(t_i + h/2, w_i + k_2/2) \\
    k_4 &= hf(t_i + h, w_i + k_3) \\
    w_{i+1} &= w_i + k_1/6 + k_2/3 + k_3/3 + k_4/6
\end{align*}
$$

with a step size of $h = 1$ to approximate the value of $y(1)$, given that $y' = y$, $y(0) = 1$.

10. Use Taylor’s method of order 2 with a step size of $h = 0.5$ to estimate $y(1)$, given that $y' = y$, $y(0) = 1$. 

2