Math 104: Introduction to Analysis
Midterm March 20th, 2002
Weingart

Name: __________________________

Signature: _______________________

There are 9 problems on this midterm worth 100 points of 400 for the class in total. The first 5 problems are each worth 8 points for the correct answer, whereas the last 4 problems are more difficult and worth 15 points each. You must show your work to get any credit for the last 4 problems. Successful midterm!
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**Problem 1:** (8 points)
Recall that 0 is called a limiting point of a sequence \((s_n)_{n \geq 1}\) if for every \(\varepsilon > 0\) there are infinitely many \(n \in \mathbb{N}\) with \(|s_n| < \varepsilon\). If however 0 is NOT a limiting point for a sequence \((s_n)_{n \geq 1}\), what do you conclude?

- [ ] There is some \(\varepsilon > 0\) such that \(|s_n| \geq \varepsilon\) for all but finitely many \(n \in \mathbb{N}\).
- [ ] There is some \(\varepsilon > 0\) such that \(|s_n| \geq \varepsilon\) for infinitely many \(n \in \mathbb{N}\).
- [ ] For all \(\varepsilon > 0\) there are infinitely many \(n \in \mathbb{N}\) with \(|s_n| \geq \varepsilon\).

**Problem 2:** (8 points)
Every rational number can be written in lowest possible terms \(\frac{p}{q}\), so that \(p\) and \(q\) have no common divisor and \(q > 0\). Consider the function \(f : \mathbb{Q} \cap [0, 1] \rightarrow \mathbb{R}\) defined on the rational numbers \(r \in [0, 1]\) by writing \(r = \frac{p}{q}\) in lowest possible terms and setting \(f(r) := \frac{1}{q}\).

- [ ] There are different continuous functions \(g : [0, 1] \rightarrow \mathbb{R}\) extending \(f\) such that \(g(r) = f(r)\) for all rational numbers \(r\) in \([0, 1]\).
- [ ] There is exactly one continuous function \(g : [0, 1] \rightarrow \mathbb{R}\) extending \(f\) such that \(g(r) = f(r)\) for all rational numbers \(r\) in \([0, 1]\).
- [ ] There is no continuous function \(g : [0, 1] \rightarrow \mathbb{R}\) extending \(f\) such that \(g(r) = f(r)\) for all rational numbers \(r\) in \([0, 1]\).

**Problem 3:** (8 points)
Consider an interval \(I \subset \mathbb{R}\) and some continuous function \(f : I \rightarrow \mathbb{R}\) defined on \(I\). Which of the following statements is true?

- [ ] If \(I = [a, b]\) is a closed interval then \(f(I) \subset \mathbb{R}\) is a closed interval for every continuous function \(f\) defined on \(I\).
- [ ] If \(I = [a, \infty)\) is a closed interval then \(f(I) \subset \mathbb{R}\) is a closed interval for every continuous function \(f\) defined on \(I\).
- [ ] If \(I = (a, b)\) is an open interval then \(f(I) \subset \mathbb{R}\) is an open interval for every continuous function \(f\) defined on \(I\).
Problem 4: (8 points)
This impressive list of names and well-sounding statements connected with them contains
one flawed reformulation, namely?

☐ By the Theorem of Heine–Borel every closed subset $A$ of $\mathbb{R}$ is compact
   or unbounded.
☐ By Banach’s Fixed Point Theorem every contraction $f : S \rightarrow S$ of a
   Cauchy–complete metric space $S$ has a unique fixed point.
☐ By the Theorem of Bolzano–Weierstraß every bounded sequence $(s_n)_{n \geq 1}$
   of real numbers has a convergent subsequence.

Problem 5: (8 points)
Calculating the radius of convergence of a power series can be tricky. Unluckily I was careless
and made an error in my calculations, please find it:

☐ The radius of convergence of the power series $\sum_{m=0}^{\infty} (m+1) x^m$ is $R = 1$.
☐ The radius of convergence of the power series $\sum_{m=0}^{\infty} \left(\frac{1}{4}\right)^m x^{2m}$ is $R = 2$.
☐ The radius of convergence of the power series $\sum_{m=0}^{\infty} \left(\frac{2m}{m}\right) x^m$ is $R = 4$.

Problem 6: (15 points)
Formulate the completeness axiom of the real numbers and write down the definition of the
limit of a convergent sequence of real numbers.
Problem 7: (15 points)
Show that the two functions \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) and \( g : \mathbb{R}^2 \rightarrow \mathbb{R} \) defined by \( f(x, y) := x \) and \( g(x, y) := y \) are continuous. Conclude that the function \( f^2 g \) defined by \( (f^2 g)(x, y) := x^2 y \) is continuous as well.

Problem 8: (15 points)
Consider a closed subset \( A \subset S \) of a metric space \( S \), in other words its complement \( S \setminus A \) is open. Prove explicitly that every convergent sequence \( (a_n)_{n \geq 1} \) of elements \( a_n \in A \) converges to a limit \( \lim_{n \to \infty} a_n \) in \( A \).
Problem 9: (15 points)
Consider a closed interval \([0, 1] \subset \mathbb{R}\) and the set of all continuous functions on \([0, 1]\\):

\[
C^0([0, 1]) := \{ f : [0, 1] \to \mathbb{R} \mid f \text{ is continuous} \}.
\]

Verify the axioms of a metric space for the following distance function on \(C^0([0, 1])\\):

\[
\text{dist}(f, g) := \sup \{ |f(x) - g(x)| \mid x \in [0, 1] \} \quad f, g \in C^0([0, 1]).
\]

Note that for fixed \(f, g \in C^0([0, 1])\) the supremum in the definition of \(\text{dist}(f, g)\) is always a real number and never \(+\infty\). Why?