MATH 185 - FINAL EXAM  Spr 02
L. Evans

INSTRUCTIONS: Answer each question on a separate sheet of paper, and write your name on each page. Each problem counts equally. Good luck.

Problem #1. Calculate the residue of

\[ \frac{z^5}{z^4 + 16} \]

at \( z_0 = \sqrt{2}(1 + i) \).

Problem #2. Show that

\[ \left| \frac{2z - i}{2 + iz} \right| = 1 \]

if \( |z| = 1 \).

Problem #3. Compute the first 4 terms of the Laurent expansion for

\[ \frac{1}{e^z - 1} \]

around \( z_0 = 0 \).

Problem #4. Find complex numbers \( a, b, c, d \) so that the linear fractional transformation

\[ T(z) = \frac{az + b}{cz + d} \]

satisfies

\[ T(-1) = \infty, \quad T(0) = 1, \quad T(i) = i. \]

Problem #5. Compute

\[ \int_0^{2\pi} \frac{1}{3 + \cos\theta} d\theta. \]

Problem #6. Assume \( a, b > 0 \) and calculate

\[ \int_{-\infty}^{\infty} \frac{\cos bx - \cos ax}{x^2} dx. \]
Problem #7. (i) Find the image of the curve drawn in the picture under the mapping

\[ f(z) = z + \frac{1}{z}. \]

In particular, plot the images \( A', B', C', D', E' \) of the points \( A, B, C, D, E \).

(ii) What is \( f'(z) \) at the points \( z = B, D \)? Is the mapping conformal at these points? Explain your answer.

Problem #8. (i) How many zeros (counting multiplicity) does the polynomial

\[ z^8 + 3z^5 + 8z^2 + z + 1 \]

have within the region \( |z| < 1 \)?

(ii) How many zeros does this polynomial have within the region \( |z| < 2 \)?

Explain carefully how you reached your answers.

Problem #9. Let \( C \) be a simple closed curve, which is the boundary of a region \( R \). Assume \( f \) is analytic within \( R \) and \( f' \) is continuous on \( R \cup C \).

Use Green’s Theorem to prove Cauchy’s Theorem:

\[ \int_C f \, dz = 0. \]

Problem #10. Assume that \( f \) is analytic within the ring \( R_1 < |z| < R_2 \). Start with Cauchy’s integral formula and derive the Laurent expansion:

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n}. \]

Write out integral formulas for the coefficients \( a_n, b_n \).
Problem #11. Suppose that $f$ is analytic within a domain $D$, and $|f(z)| = 1$ for all $z \in D$.

Prove that $f$ is constant.

Problem #12. Prove the Casorati–Weierstrass Theorem:

Assume $f$ is analytic for $0 < |z - z_0| < 1$, and suppose that $z_0$ is an essential singularity of $f$. Let $w_0$ be any complex number. Then for each $\epsilon > 0$ and each $\delta > 0$, the inequality

$$|f(z) - w_0| < \epsilon$$

is satisfied for some point $z$ in the deleted neighborhood $0 < |z - z_0| < \delta$.

(Hint: You may quote any relevant theorem on removable singularities.)