1 (1/6) Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic nonconstant function on $\mathbb{C}$ and define the real valued function $m_f : [0, +\infty) \to \mathbb{R}$ by $m_f(t) = \sup\{|f(z)| : |z| = t\}$. Show that $m_f$ is strictly increasing on $[0, +\infty)$.
2 (1/6) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that there exist uniquely determined entire functions $f_1$ and $f_2$ satisfying the following two conditions:

(a) $f = f_1 + if_2$ on $\mathbb{C}$ and
(b) $f_1$ and $f_2$ are real valued on $\mathbb{R}$. 
3 (1/6) Let $U$ be the open set $U = \mathbb{C} \setminus \{0, -1, -2, -3, \ldots\}$, i.e., remove zero and the negative integers from $\mathbb{C}$. Assume $f : U \to \mathbb{C}$ is holomorphic, $f(1) = 1$ and $zf(z) = f(z + 1)$ for all $z \in U$. Show that $f$ has a simple pole at each point $m \in \{0, -1, -2, -3, \ldots\}$ and that the residue of $f$ at $m = -n$ is given by

$$\text{Res}_{-n}(f) = \frac{(-1)^n}{n!}$$
4 (1/6) Let \( n \) be a positive integer. Determine the number of zeros of the function

\[ g(z) = 2(z - 1)^n - e^{-z} \]

inside the open disk \( U(1, 1) \) and show that all the zeros are of order 1.

[Remember: If \( z_0 \) is a zero of \( g \) then it is enough to write \( g(z) = (z - z_0)h(z) \) with \( h(z_0) \neq 0 \) in order to show that \( z_0 \) is of order 1.]
5 (1/6) Let \( m \in \mathbb{N} \). Calculate the integral

\[
\int_{-\infty}^{+\infty} \frac{dx}{1 + x + x^2 + \cdots + x^{2m}} = \int_{-\infty}^{+\infty} \frac{1 - x}{1 - x^{2m+1}} \, dx.
\]
6 (1/6) Describe all the automorphisms of the first quadrant

\[ Q_1 = \{ z : \text{Re}(z) > 0, \text{Im}(z) > 0 \} \]

in terms of the 2x2 real matrices with determinant 1. [Remember: All the automorphisms of the upper half plane are of the form \( \frac{az+b}{cz+d} \) where \( a, b, c, d \) are real with \( ac - bd = 1 \).]