MATH 121B: MIDTERM 2, SPRING, 2000

Total score: 100 points.

Problem 1.
(i) (10 points) Directly from the definition of the $\Gamma$ function, show that for $p > 0$ real, $\Gamma(p+1) = p\Gamma(p)$.
(ii) (10 points) Find $\int_0^1 x(\ln \frac{1}{x})^{3/2} \, dx$. (Hint: change variables. You may use that $\Gamma(1/2) = \sqrt{\pi}$.)

Problem 2. (25 points) Using Frobenius' method of generalized (fractional) power series, solve the ODE

$$9z^2y'' + 2(1 + z^2)y = 0.$$ 

It suffices to find the first three non-zero terms of two linearly independent solutions of the ODE.

Problem 3. Consider a sector of a circular plate of radius $a = 2$ and angle $\pi/3$. Suppose that its straight sides are kept at 0 temperature, and the curved side at temperature $20\sin(9\theta)$. Find the steady state temperature inside the sector as follows.

(i) (17 points) Separate variables in polar coordinates. Recall that, in two dimensions, the Laplacian in polar coordinates $(r, \theta)$ is given by

$$\Delta_{xy}u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$ 

Use this and the homogeneous boundary conditions to find the general form of the solution.

(ii) (8 points) Find the coefficients of the separated solutions by making sure that the inhomogeneous boundary condition is satisfied.

Problem 4. Consider a cylinder of radius 1, height 1. Take the $z$ axis as the axis of symmetry, and use cylindrical coordinates below. Suppose that the bottom, $z = 0$, and the side, $r = 1$, are kept at zero temperature, and the top, $z = 1$, at temperature $100r^2 \cos 2\theta$ degrees. We want to find the steady state temperature in the cylinder.

(i) (15 points) Separate variables to find the general solution that satisfies all of the homogeneous boundary conditions. You may use that eigenfunctions $v$ of the Laplacian on the unit disk satisfying Dirichlet boundary condition, $v(1, \theta) = 0$, are

$$v = J_n \left( k_{nm} r \right) \left( A_{nm} \cos(n\theta) + B_{nm} \sin(n\theta) \right),$$

with $\Delta_{xy}v = -k_{nm}^2 v$; here $k_{nm}$ is the $m$th zero of $J_n$.

(ii) (15 points) Find the coefficients in the expansion, hence the solution of the original problem. Make sure that your final formula contains no integrals. You may use that the Bessel functions $J_p$ satisfy $\frac{d}{dx} \left[ x^p J_p(x) \right] = x^p J_{p-1}(x)$. You may also use that $\int_0^1 J_n(k_{nm} r)^2 r \, dr = J_{n+1}(k_{nm})^2 / 2$. 