[1] 10 points
Solve
\[ y''(t) + y(t) = \sin(t) \quad ; \quad y(0) = 0, \ y'(0) = 0. \]

[2] 5 points
Find the Laplace transform of
\[ f(t) = \begin{cases} 
0 & \text{if } 0 \leq t < 2 \\
3e^{-(t-2)} & \text{if } 2 \leq t 
\end{cases} \]
[3] \hspace{1cm} 5 \times 2 \text{ points}

Let \( f(x) \) and \( g(x) \) be absolutely integrable functions on \( \mathbb{R} \). Let \( \mathcal{F}[f] \) denote the Fourier transform of \( f(x) \).

(i) What formula defines the Fourier transform of \( f(x) \)?

(ii) What formula defines \( f \ast g \)?

(iii) Express \( \mathcal{F}[f'(x)] \) in terms of \( \mathcal{F}[f] \).

(iv) Express \( \mathcal{F}[f \ast g] \) in terms of \( \mathcal{F}[f] \) and \( \mathcal{F}[g] \).

(v) Express \( f \) in terms of \( \mathcal{F}[f] \).
[4]  

5 points

Evaluate

\[ \oint_C \frac{dz}{z}, \]

where \( C \) is the circle of radius 2 centered at 0, oriented counterclockwise.
[5] 10 points

Let \( z = \sqrt{3} - i \).

(i) Write \( z \) as \( re^{i\theta} \), i.e., find \( r \) and \( \theta \).

(ii) What is \( z^3 \)?

(iii) What are \( \Re(1/z) \) and \( \Im(1/z) \), the real and imaginary parts of \( 1/z \)?

(iv) What is \( \Im(\overline{z}) \)?
[6] 5 points

Find the radius of convergence of \( \sum_{n=1}^{\infty} 2n z^{2n-1} \).

[7] 5 points

What does \( \sum_{n=1}^{\infty} 2n z^{2n-1} \) equal where it converges?
[8] 10 points

Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be analytic, and let \( u(x, y) = \Re(f(x + iy)) \).

Show that
\[
\frac{\partial^2}{\partial x^2} u(x, y) + \frac{\partial^2}{\partial y^2} u(x, y) = 0.
\]

[Hint: Use the Cauchy-Riemann equations]
[9] Statement of definitions  2 x 5 points
(i) Let $S$ be a set of real numbers. Carefully define $\sup S$, the supremum (or least upper bound) of $S$.

(ii) Let $f : D \rightarrow \mathbb{C}$ be a complex-valued function on a domain $D$, and for each $n \in \mathbb{N}$, let $f_n : D \rightarrow \mathbb{C}$. By definition, $\{f_n\}$ converges uniformly to $f$ (on $D$) if and only if . . .
[10] 10 points

Solve Schrödinger's equation on $\mathbb{R}$ using the Fourier Integral:

$$i \frac{\partial}{\partial t} \psi(x, t) = -\frac{\partial^2}{\partial x^2} \psi(x, t)$$
$$\psi(x, 0) = f(x).$$
Let $f: D \rightarrow \mathbb{C}$ be defined on an open domain $D$ of the complex plane, and let $w$ be a point in $D$. Recall that

$$\lim_{z \to w} f(z) = c,$$

if and only if the following condition is met: For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z) - c| < \epsilon$ if $|z - w| < \delta$.

Prove one direction — your choice, either “if” or “only if” — of the following theorem. For extra credit, prove both directions.

**Theorem:**

$$\lim_{z \to w} f(z) = c$$

if and only if $\{f(z_n)\}$ converges to $c$ whenever $\{z_n\}$ converges to $w$. 