Math 128A - Final Exam  
Spring 2000 – Nate Brown

1) (10pts) Assume $N$ is a constant, $N$ is a function of $h$ and $M = N(h) + h^2 + h^3 + h^4 + \cdots$. Use Richardson’s extrapolation to find a function $\tilde{N}$ and constants $K_3, K_4, \ldots$ such that $M = \tilde{N}(h) + K_3h^3 + K_4h^4 + \cdots$. (Hint: Replace $h$ with $2h$.)

2) (10pts) Use a divided difference table to find a cubic polynomial whose graph passes through the points $(-1, 2), (0, 1), (1, 0)$ and $(2, 5)$.

3) (20pts) Use Taylor’s theorem to prove that if $f \in C^3[a, b]$, $x_0 \in [a, b]$ and $h > 0$ is such that $x_0 + h \in [a, b]$ then there exists $\xi \in [x_0, x_0 + h]$ such that

$$f'(x_0) = \frac{1}{h} \left( f(x_0 + h) - f(x_0) \right) - \frac{1}{2} \frac{f''(\xi)}{h}.$$

4) Let $f(x) = x^2 + (1/x)$. Let $Q_n$ be the Newton polynomial and $H_{2n+1}$ be the Hermite polynomial which interpolate $f$ at distinct points $x_0, \ldots, x_n \in [2, 3]$.

a) (10pts) Find an $n$ such that $|f(x) - Q_n(x)| < 10^{-6}$ for all $x \in [2, 3]$.

b) (10pts) Find an $n$ such that $|f(x) - H_{2n+1}(x)| < 10^{-6}$ for all $x \in [2, 3]$.

5) (20pts) Use the first LaGrange interpolating polynomial (at the endpoints) to prove the simple Trapezoidal rule. That is, prove that if $f \in C^2[a, b]$ then there exists $\xi \in [a, b]$ such that

$$\int_a^b f(x)dx = \frac{b-a}{2} (f(a) + f(b)) + \frac{(b-a)^3}{12} f''(\xi).$$

(Use the identities $\int_a^b x dx = \frac{b^2-a^2}{2}$ and $\int_a^b (x-a)(x-b)dx = \frac{(b-a)^3}{3}$.)

6) Let $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{pmatrix}$.

a) (10pts) Find elementary matrices $A_1, A_2$ such that $A = A_2A_1$ is upper triangular.

b) (10pts) Let $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Find a vector $\vec{c}$ and an upper triangular matrix $A$ such that if $\vec{x}$ is a solution to $A\vec{x} = \vec{c}$ then $\vec{x}$ is also a solution to $A\vec{x} = \vec{b}$.
7) a)(10pts) If \( T \in M_n(\mathbb{R}) \), \( \mathbf{c} \in \mathbb{R}^n \) describe an iterative technique for solving the fixed point type problem \( \mathbf{x} = T \mathbf{x} + \mathbf{c} \).

b)(10pts) Describe the Jacobi method for solving linear systems.

c)(5pts) Will the Jacobi method converge if \( A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix} \)?

8) (10pts) Assume \( A \in M_n(\mathbb{R}) \) is invertible and we wish to solve the linear system \( A \mathbf{x} = \mathbf{b} \). In general, the Gauss-Seidel method applied to this system will not converge. Find a linear system \( B \mathbf{x} = \mathbf{c} \) with the properties that i) if \( \mathbf{x} \) is a solution to the equation \( B \mathbf{x} = \mathbf{c} \) then \( \mathbf{x} \) is also a solution to the equation \( A \mathbf{x} = \mathbf{b} \) and ii) the Gauss-Seidel method will converge for the equation \( B \mathbf{x} = \mathbf{c} \).

9) Assume \( y(t) = y(t) \), \( a \leq t \leq b \), \( y(a) = \alpha \). Let \( h = (b-a)/n \), \( t_i = a + ih \) \((1 \leq i \leq n)\), \( w_0 = \alpha \) and \( w_{i+1} = w_i + h\phi(t_i, w_i, h), \) \( 0 \leq i \leq n - 1 \), for some function \( \phi \).

a)(5pts) Define the local truncation error.

b)(5pts) Define consistency.

c)(5pts) Define convergence.

d)(20pts) If \( f(t, y) = t^2 e^t - 3t^2 y \), prove that the modified Euler method will converge. (Recall that for this method \( w_{i+1} = w_i + h/2[f(t_i, w_i) + f(t_{i+1}, w_i + h f(t_i, w_i))] \).)

10) a)(5pts) Let \{y_n\} be a convergent sequence of numbers with limit \( y \). Define what it means for \{y_n\} to converge quadratically (i.e. with order of convergence 2).

b)(25pts) Assume \( f \in C^2[a, b], \) \( p \in [a, b], \) \( f(p) = 0 \) and \( f'(p) \neq 0 \). Prove that there exists \( \delta > 0 \) such that for any initial guess \( x_0 \in [p-\delta, p+\delta] \), Newton's method will converge (at least) quadratically to \( p \).