1. Let \( \alpha(s) = (r(s), 0, z(s)) \) be a unit speed curve in the \( xz \)-plane such that \( r > 0 \), and let \( \tilde{\gamma}(s, \theta) = (r(s) \cos \theta, r(s) \sin \theta, z(s)) \) describe the simple surface (coordinate patch) \( M \) (where \( -\pi < \theta < \pi \)) obtained by revolving \( \alpha \) around the \( z \)-axis. Show the following (you are allowed to make use of everything that precedes an item to do that item):

(a) Show that the metric coefficients \( \{ g_{ij} \} \) of \( M \) are given by the matrix \( \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \). (8 pts.)

(b) Define a geodesic on a surface. Use it to show that each \( s \)-curve on \( M \) (meridian) is a geodesic. (8 pts.)

(c) With \( \tilde{x}_s = \frac{\partial \tilde{x}}{\partial s} \), \( \tilde{x}_\theta = \frac{\partial \tilde{x}}{\partial \theta} \) as usual, compute the matrix of the second fundamental form:

\[
\begin{pmatrix}
\langle \tilde{\gamma}_s, \tilde{\gamma}_s \rangle & \langle \tilde{\gamma}_s, \tilde{\gamma}_\theta \rangle \\
\langle \tilde{\gamma}_\theta, \tilde{\gamma}_s \rangle & \langle \tilde{\gamma}_\theta, \tilde{\gamma}_\theta \rangle
\end{pmatrix}
\] (14 pts.)
(d) Note that as in (c), compute the matrix of the Weingarten map \( \mathbf{L} \) relative to \( \{ \mathbf{x}, \mathbf{y} \} \). (14 pts.)

(e) Show that the Gaussian curvature \( K \) and the mean curvature \( H \) satisfy
\[
K = -\frac{\mathbf{r} \cdot \mathbf{r}}{r^2}
\]
and
\[
H = \frac{1}{2n} \left( n \mathbf{r} \cdot \mathbf{z} - r \mathbf{z} \cdot \mathbf{z} + \mathbf{z} \right).
\] (8 pts.)

(f) Construct a surface of Gaussian curvature \(-1\). (8 pts.)

2. (a) If every point of a simple surface \( M \) is an umbilic, show that \( M \) has constant principal curvature. (10 pts.)

(b) Use (a) to show that if a simple surface \( M \) has positive Gaussian curvature and each point of \( M \) is an umbilic, then \( M \) is part of a sphere. (10 pts.)

[Recall that every simple surface is connected.]

3. Let \( \mathcal{R} \) be an "annulus" region (as shown) on a simple surface where both the inner and outer boundary curves are smooth. Starting with the Gauss-Bonnet theorem for triangles, compute
\[
\int_{\mathcal{R}} K \mathbf{d}A + \int_{\partial \mathcal{R}} \kappa_g \mathbf{d}s
\]
\((K = \text{Gaussian curvature}, \kappa_g = \text{geodesic curvature})\). (20 pts.)