Preliminary Exam - Fall 1999

Problem 1 *Let V and W be finite dimensional vector spaces, let X be a subspace of W, and let $T : V \to W$ be a linear map. Prove that the dimension of $T^{-1}(X)$ is at least* $\dim V - \dim W + \dim X$.

Problem 2 *Let $E_1, E_2, \ldots$ be nonempty closed subsets of a complete metric space (X, d) with $E_{n+1} \subset E_n$ for all positive integers n, and such that* $\lim_{n\to\infty} \operatorname{diam}(E_n) = 0$, *where* $\operatorname{diam}(E)$ *is defined to be*

$$\sup\{d(x, y) \mid x, y \in E\}.$$

Prove that $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$.

Problem 3 *Let R be a ring with identity element. Suppose that $\mathfrak{I}_1$, $\mathfrak{I}_2$, $\ldots$, $\mathfrak{I}_n$ are left ideals in R such that $R = \mathfrak{I}_1 \oplus \mathfrak{I}_2 \oplus \cdots \oplus \mathfrak{I}_n$ (as additive groups). Prove that there are elements $u_i \in \mathfrak{I}_i$ such that for any elements $a_i \in \mathfrak{I}_i$, $a_i u_i = a_i$ and $a_i u_j = 0$ if $j \neq i$.*

Problem 4 *Let the rational function f in the complex plane have no poles for $\Im z \geqslant 0$. Prove that*

$$\sup\{|f(z)| \mid \Im z \geqslant 0\} = \sup\{|f(z)| \mid \Im z = 0\}.$$

Problem 5 *Let M_n be the vector space of real $n \times n$ matrices, identified in the usual way with the Euclidean space $\mathbb{R}^{n^2}$. (Thus, the norm of a matrix $X = (x_{jk})_{j,k=1}^n$ in M_n is given by $\|X\|^2 = \sum_{j,k=1}^n x_{jk}^2$.) Define the map f of M_n into M_n by $f(X) = X^2$. Determine the derivative Df of f.*

Problem 6 *Let $T : V \to V$ be a linear operator on an n dimensional vector space V over a field $\mathbf{F}$. Prove that T has an invariant subspace W other than $\{0\}$ and V if and only if the characteristic polynomial of T has a factor $f \in \mathbf{F}[t]$ with $0 < deg\ f < n$.*

Problem 7 *Let G be a finite group acting transitively on a set X of size at least 2. Prove that some element g of G acts without fixed points.*

Problem 8 *Evaluate the integral*

$$I = \frac{1}{2\pi i} \int_{|z|=1} \frac{(z+2)^2}{z^2(2z-1)} dz\,,$$

where the direction of integration is counterclockwise.

Problem 9 *Describe all three dimensional vector spaces* V *of* C^∞ *complex valued functions on* $\mathbb{R}$ *that are invariant under the operator of differentiation.*

Problem 10 *Let* f *be a continuous real valued function on* $[0,\infty)$ *such that* $\lim_{x\to\infty} f(x)$ *exists (finitely). Prove that* f *is uniformly continuous.*

Problem 11 *Let* V *be a finite dimensional vector space over a field* $\mathbf{F}$*, and let* A *and* B *be diagonalizable linear operators on* V *such that* $AB = BA$*. Prove that* A *and* B *are simultaneously diagonalizable, in other words, that there is a basis for* V *consisting of eigenvectors of both* A *and* B*.*

Problem 12 *Let* $A = \{0\} \cup \{1/n \mid n \in \mathbb{Z}, n > 1\}$*, and let* $\mathbb{D}$ *be the open unit disc in the complex plane. Prove that every bounded holomorphic function on* $\mathbb{D} \setminus A$ *extends to a holomorphic function on* $\mathbb{D}$*.*

Problem 13 *Let* $\mathbf{K}$ *be the field* $\mathbb{Q}(\sqrt[10]{2})$*. Prove that* $\mathbf{K}$ *has degree* 10 *over* $\mathbb{Q}$*, and that the group of automorphisms of* $\mathbf{K}$ *has order* 2*.*

Problem 14 *Show that every infinite closed subset of* $\mathbb{R}^2$ *is the closure of a countable set.*

Problem 15 *Let* A *be an* $n\times n$ *complex matrix such that* $\operatorname{tr} A^k = 0$ *for* $k = 1, \ldots, n$*. Prove that* A *is nilpotent.*

Problem 16 *For* $0 < a < b$*, evaluate the integral*

$$I = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|ae^{i\theta} - b|^4} d\theta\,\cdot$$

Problem 17 *Show that a group* G *is isomorphic to a subgroup of the additive group of the rationals if and only if* G *is countable and every finite subset of* G *is contained in an infinite cyclic subgroup of* G*.*

Problem 18 *Let* f *and* g *be continuous real valued functions on* $\mathbb{R}$ *such that* $\lim_{|x|\to\infty} f(x) = 0$ *and* $\int_{-\infty}^{\infty} |g(x)|dx < \infty$. *Define the function* h *on* $\mathbb{R}$ *by*

$$h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy\,.$$

Prove that $\lim\limits_{|x|\to\infty} h(x) = 0$.