Preliminary Exam - Fall 1998

Problem 1 *Let M be a metric space with metric d. Let C be a compact subset of M, and let $(U_\alpha)_{\alpha\in I}$ be an open cover of C. Show that there exists $\epsilon > 0$ such that, for every $p \in C$, the open ball $B(p,\epsilon)$ is contained in at least one of the sets U_α.*

Problem 2 *Find a function $y(x)$ such that $y^{(4)} + y = 0$ for $x \geqslant 0$, $y(0) = 0$, $y'(0) = 1$ and $\lim_{x\to\infty} y(x) = \lim_{x\to\infty} y'(x) = 0$.*

Problem 3 *Prove that for any real $\alpha > 1$, $\displaystyle\int_0^\infty \frac{dx}{1+x^\alpha} = \frac{\pi/\alpha}{\sin(\pi/\alpha)\cdot}$*

Problem 4 *Let f be analytic in the closed unit disc, with$f(-\log 2) = 0$ and $|f(z)| \leqslant |e^z|$ for all z with $|z| = 1$. How large can $|f(\log 2)|$ be?*

Problem 5 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that*

1. *the function g defined by $g(x,y) = f(x+y) - f(x) - f(y)$ is* bilinear,
2. *for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $f(tx) = t^2 f(x)$.*

Show that there is a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x) = \langle x, Ax\rangle$ where $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$ (in other words, f is a quadratic form).

Problem 6 *Let A and B be linear transformations on a finite dimensional vector space V. Prove that* $\dim\ker(AB) \leqslant \dim\ker A + \dim\ker B$.

Problem 7 *A real symmetric $n\times n$ matrix A is called* positive semi-definite *if $x^t Ax \geqslant 0$ for all $x \in \mathbb{R}^n$. Prove that A is positive semi-definite if and only if* $\operatorname{tr} AB \geqslant 0$ *for every real symmetric positive semi-definite $n\times n$ matrix B.*

Problem 8 *Let R be a finite ring with identity. Let a be an element of R which is not a zero divisor. Show that a is invertible.*

Problem 9 *Suppose that G is a finite group such that every Sylow subgroup is normal and abelian. Show that G is abelian.*

Problem 10 *Let f be a real function on $[a, b]$. Assume that f is differentiable and that f' is Riemann integrable. Prove that*

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Problem 11 *Find the minimal value of the areas of hexagons circumscribing the unit circle in $\mathbb{R}^2$.*
Note: See also Problem **??**.

Problem 12 *Let $\varphi(x, y)$ be a function with continuous second order partial derivatives such that*

1. $\varphi_{xx} + \varphi_{yy} + \varphi_x = 0$ *in the punctured plane* $\mathbb{R}^2 \setminus \{0\}$,
2. $r\varphi_x \to \dfrac{x}{2\pi r}$ *and* $r\varphi_y \to \dfrac{y}{2\pi r}$ *as* $r = \sqrt{x^2 + y^2} \to 0$.

Let C_R be the circle $x^2 + y^2 = R^2$. Show that the line integral

$$\int_{C_R} e^x(-\varphi_y\,dx + \varphi_x\,dy)$$

is independent of R, and evaluate it.

Problem 13 *Let f be an entire function. Define $\Omega = \mathbb{C} \setminus (-\infty, 0]$, the complex plane with the ray $(-\infty, 0]$ removed. Suppose that for all $z \in \Omega$, $|f(z)| \leqslant |\log z|$, where $\log z$ is the principal branch of the logarithm. What can one conclude about the function f?*

Problem 14 *Let $z_1, \ldots, z_n$ be distinct complex numbers, and let$a_1, \ldots, a_n$ be nonzero complex numbers such that $S_p = \sum_{j=1}^n a_j z_j^p = 0$ for $p = 0, 1, \ldots, m-1$ but $S_m \neq 0$. Here $1 \leqslant m \leqslant n-1$. How many zeros does the rational function $f(z) = \sum_{j=1}^n \frac{a_j}{z - z_j}$ have in $\mathbb{C}$? Why is $m \geqslant n$ impossible.*

Problem 15 *Let A and B be $n \times n$ matrices. Show that the eigenvalues of AB are the same as the eigenvalues of BA.*

Problem 16 *Let B be a 3×3 matrix whose null space is 2-dimensional, and let $\chi(\lambda)$ be the characteristic polynomial of B. For each assertion below, provide either a proof or a counterexample.*

1. λ^2 *is a factor of* $\chi(\lambda)$.

2. *The trace of* B *is an eigenvalue of* B.

3. B *is diagonalizable.*

Problem 17 *Let* $\mathbf{F}$ *be a finite field with* q *elements. Denote by* $GL_n(\mathbf{F})$ *the group of invertible* $n \times n$ *matrices with entries if* $\mathbf{F}$. *What is the order of this group?*

Problem 18 *Show that the field* $\mathbb{Q}(t_1, \ldots, t_n)$ *of rational functions in* n *variables over the rational numbers is isomorphic to a subfield of* $\mathbb{R}$.