Preliminary Exam - Fall 1997

Problem 1 *Define a sequence of real numbers* (x_n) *by*

$$x_0 = 1, \qquad x_{n+1} = \frac{1}{2 + x_n} \quad \text{for} \quad n \geqslant 0.$$

Show that (x_n) *converges, and evaluate its limit.*

Problem 2 *Let* f *be a real valued function that is differentiable on an open interval containing* $[a, b]$. *Prove that if* $f'(a) < 0$ *and* $f'(b) > 0$ *then there is a point* $c \in (a, b)$ *such that* $f'(c) = 0$.

Problem 3 *Let* f *be an entire function such that, for all* z, $|f(z)| = |\sin z|$. *Prove that there is a constant* C *of modulus* 1 *such that* $f(z) = C \sin z$.

Problem 4 *Evaluate the integral*

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^{2n}}$$

where $n > 0$ *is an integer.*

Problem 5 *Let* $\mathbb{D} = \{z \mid |z| < 1\}$, *the open unit disc in the complex plane. Suppose that* $f : \mathbb{D} \to \mathbb{D}$ *is analytic, and that there exist two distinct points* $a, b \in \mathbb{D}$ *with* $f(a) = a$, $f(b) = b$. *Prove that* $f(z) = z$ *for all* $z \in \mathbb{D}$.

Problem 6 *Let* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *be distinct real numbers. Show that the* n *exponential functions* $e^{\alpha_1 t}, e^{\alpha_2 t}, \ldots, e^{\alpha_n t}$ *are linearly independent over the real numbers.*

Problem 7 *Define the* index *of a real symmetric matrix* A *to be the number of strictly positive eigenvalues of* A *minus the number of strictly negative eigenvalues. Suppose* A, *and* B *are real symmetric* $n \times n$ *matrices such that* $x^t A x \leqslant x^t B x$ *for all* $n \times 1$ *matrices* x. *Prove the the index of* A *is less than or equal to the index of* B.

Problem 8 *Suppose H_i is a normal subgroup of a group G for$1 \leqslant i \leqslant k$, such that $H_i \cap H_j = \{1\}$ for $i \neq j$. Prove that G contains a subgroup isomorphic to $H_1 \times H_2 \times \cdots \times H_k$ if $k = 2$, but not necessarily if $k \geqslant 3$.*

Problem 9 *Prove that if p is prime then every group of order p^2 is abelian.*

Problem 10 *Prove that for all $x > 0$, $\sin x > x - \dfrac{x^3}{6}$.*

Problem 11 *Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable, and suppose that for all $x \in \mathbb{R}$, $|f(x)| \leqslant 1$ and $|f''(x)| \leqslant 1$. Prove that $|f'(x)| \leqslant 2$ for all $x \in \mathbb{R}$.*

Problem 12 *A map $f : \mathbb{R}^m \to \mathbb{R}^n$ is* proper *if it is continuous and $f^{-1}(B)$ is compact for each compact subset B of $\mathbb{R}^n$; f is* closed *if it is continuous and $f(A)$ is closed for each closed subset A of $\mathbb{R}^m$.*

1. *Prove that every proper map $f : \mathbb{R}^m \to \mathbb{R}^n$ is closed.*

2. *Prove that every one-to-one closed map $f : \mathbb{R}^m \to \mathbb{R}^n$ is proper.*

Problem 13 *Conformally map the region inside the disc given by$\{z \in \mathbb{C} \mid |z-1| \leqslant 1\}$ and outside the disc $\{z \in \mathbb{C} \mid |z - \frac{1}{2}| \leqslant \frac{1}{2}\}$ onto the upper half-plane.*

Problem 14 *Evaluate the integral*

$$\int_{-\infty}^{\infty} \frac{\cos kx}{1 + x + x^2}\, dx$$

where $k \geqslant 0$.

Problem 15 *Let $M_{n\times n}(\mathbf{K})$ be the vector space of $n \times n$ matrices over a field $\mathbf{K}$. Find the dimension of the subspace of $M_{n\times n}(\mathbf{K})$ spanned by $\{XY - YX \mid X, Y \in M_{n\times n}(\mathbf{K})\}$.*

Problem 16 *Prove that if A is a 2×2 matrix over the integers such that $A^n = I$ for some strictly positive integer n, then $A^{12} = I$.*

Problem 17 *A group G is generated by two elements a, b, each of order 2. Prove that G has a cyclic subgroup of index 2.*

Problem 18 *A finite abelian group G has the property that for each positive integer n the set $\{x \in G \mid x^n = 1\}$ has at most n elements. Prove that G is cyclic, and deduce that every finite field has cyclic multiplicative group.*