Problem 1 Let $M$ be the set of real valued continuous functions $f$ on $[0, 1]$ such that $f'$ is continuous on $[0, 1]$, with the norm
\[
\|f\| = \sup_{0 \leq x \leq 1} |f(x)| + \sup_{0 \leq x \leq 1} |f'(x)|.
\]
Which subsets of $M$ are compact?

Problem 2 A real-valued function $f$ on a closed bounded interval $[a, b]$ is said to be upper semicontinuous provided that for every $\varepsilon > 0$ and $p \in [a, b]$, there is a $\delta = \delta(\varepsilon, p) > 0$ such that if $x \in [a, b]$ and $|x - p| < \delta$ then $f(x) < f(p) + \varepsilon$. Prove that an upper semicontinuous function is bounded above on $[a, b]$.

Problem 3 Evaluate the integral
\[
I = \int_0^\infty \frac{\sqrt{x}}{1 + x^2} \, dx.
\]

Problem 4 Does there exist a function $f$, analytic in the punctured plane $\mathbb{C} \setminus \{0\}$, such that
\[
|f(z)| \geq \frac{1}{\sqrt{|z|}}
\]
for all nonzero $z$?

Problem 5 Prove that any linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has

1. a one-dimensional invariant subspace

2. a two-dimensional invariant subspace.

Problem 6 Let $A$ and $B$ be real $2 \times 2$ matrices such that
\[
A^2 = B^2 = I, \quad AB + BA = 0.
\]
Show that there exists a real $2 \times 2$ matrix $T$ such that
\[
TAT^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad TBT^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
Problem 7 Suppose $p$ is a prime. Show that every element of $GL_2(\mathbb{F}_p)$ has order dividing either $p^2 - 1$ or $p(p - 1)$.

Problem 8 Show the denominator of $\binom{1/2}{n}$ is a power of 2 for all integers $n$.

Problem 9 For positive integers $a$, $b$ and $c$ show that

$$\gcd\{a, \text{lcm}\{b, c\}\} = \text{lcm}\{\gcd\{a, b\}, \gcd\{a, c\}\}.$$ 

Problem 10 If $f$ is a $C^2$ function on an open interval, prove that

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x).$$ 

Problem 11 Let $f$ be continuous and nonnegative on $[0, 1]$ and suppose that

$$f(t)^2 \leq 1 + 2 \int_0^t f(s)ds.$$ 

Prove that $f(t) \leq 1 + t$ for $0 \leq t \leq 1$.

Problem 12 Find the number of roots, counted with their multiplicities, of

$$z^7 - 4z^3 - 11 = 0$$

which lie between the circles $|z| = 1$ and $|z| = 2$.

Problem 13 Define $F : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$F(u, v, w) = (u + v + w, uv + vw + uw, uvw).$$

Show that $F$ is onto but not one-to-one.

Problem 14 Let $f$ be holomorphic on and inside the unit circle $C$. Let $L$ be the length of the image of $C$ under $f$. Show that

$$L \geq 2\pi|f'(0)|.$$
Problem 15 Is there a real $2 \times 2$ matrix $A$ such that

$$A^{20} = \begin{pmatrix} -1 & 0 \\ 0 & -1 - \varepsilon \end{pmatrix}.$$ 

Exhibit such an $A$ or prove there is none.

Problem 16 Let

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$ 

Show that every real matrix $B$ such that $AB = BA$ has the form

$$B = aI + bA + cA^2$$

for some real numbers $a$, $b$, and $c$.

Problem 17 Let $\mathbb{Z}[x]$ be the ring of polynomials in the indeterminate $x$ with coefficients in the ring $\mathbb{Z}$ of integers. Let $\mathfrak{I} \subset \mathbb{Z}[x]$ be the ideal generated by $13$ and $x - 4$. Find an integer $m$ such that $0 \leq m \leq 12$ and

$$(x^{26} + x + 1)^{73} - m \in \mathfrak{I}.$$ 

Problem 18 Prove that any finite group is isomorphic to

1. a subgroup of the group of permutations of $n$ objects
2. a subgroup of the group of permutations of $n$ objects which consists only of even permutations.