Preliminary Exam - Fall 1996

Problem 1 *Let M be the set of real valued continuous functions f on $[0,1]$ such that f' is continuous on $[0,1]$, with the norm*

$$\|f\| = \sup_{0\leqslant x\leqslant 1} |f(x)| + \sup_{0\leqslant x\leqslant 1} |f'(x)| \ .$$

Which subsets of M are compact?

Problem 2 *A real-valued function f on a closed bounded interval $[a,b]$ is said to be upper semicontinuous provided that for every $\varepsilon > 0$ and $p \in [a,b]$, there is a $\delta = \delta(\varepsilon, p) > 0$ such that if $x \in [a,b]$ and $|x-p| < \delta$ then $f(x) < f(p) + \varepsilon$. Prove that an upper semicontinuous function is bounded above on $[a,b]$.*

Problem 3 *Evaluate the integral*

$$I = \int_0^\infty \frac{\sqrt{x}}{1+x^2}\, dx \ .$$

Problem 4 *Does there exist a function f, analytic in the punctured plane $\mathbb{C} \setminus \{0\}$, such that*

$$|f(z)| \geqslant \frac{1}{\sqrt{|z|}}$$

for all nonzero z?

Problem 5 *Prove that any linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has*

1. *a one-dimensional invariant subspace*
2. *a two-dimensional invariant subspace.*

Problem 6 *Let A and B be real 2×2 matrices such that*

$$A^2 = B^2 = I \ , \qquad AB + BA = 0 \ .$$

Show that there exists a real 2×2 matrix T such that

$$TAT^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \qquad TBT^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ .$$

Problem 7 *Suppose* p *is a prime. Show that every element of* $GL_2(\mathbf{F}_p)$ *has order dividing either* p^2-1 *or* $p(p-1)$.

Problem 8 *Show the denominator of* $\binom{1/2}{n}$ *is a power of* 2 *for all integers* n.

Problem 9 *For positive integers* a, b *and* c *show that*

$$\gcd\{a, \operatorname{lcm}\{b,c\}\} = \operatorname{lcm}\{\gcd\{a,b\}, \gcd\{a,c\}\}.$$

Problem 10 *If* f *is a* C^2 *function on an open interval, prove that*

$$\lim_{h\to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2} = f''(x)\ .$$

Problem 11 *Let* f *be continuous and nonnegative on* $[0,1]$ *and suppose that*

$$f(t)^2 \leqslant 1 + 2\int_0^t f(s)ds\ .$$

Prove that $f(t) \leqslant 1+t$ *for* $0 \leqslant t \leqslant 1$.

Problem 12 *Find the number of roots, counted with their multiplicities, of*

$$z^7 - 4z^3 - 11 = 0$$

which lie between the circles $|z|=1$ *and* $|z|=2$.

Problem 13 *Define* $F : \mathbb{C}^3 \to \mathbb{C}^3$ *by*

$$F(u,v,w) = (u+v+w,\ uv+vw+uw,\ uvw)\ .$$

Show that F *is onto but not one-to-one.*

Problem 14 *Let* f *be holomorphic on and inside the unit circle* C. *Let* L *be the length of the image of* C *under* f. *Show that*

$$L \geqslant 2\pi|f'(0)|\ .$$

Problem 15 *Is there a real 2×2 matrix A such that*

$$A^{20} = \begin{pmatrix} -1 & 0 \\ 0 & -1-\varepsilon \end{pmatrix} \ ?$$

Exhibit such an A or prove there is none.

Problem 16 *Let*

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} .$$

Show that every real matrix B such that $AB = BA$ has the form

$$B = aI + bA + cA^2$$

for some real numbers a, b, and c.

Problem 17 *Let $\mathbb{Z}[x]$ be the ring of polynomials in the indeterminate x with coefficients in the ring $\mathbb{Z}$ of integers. Let $\mathfrak{I} \subset \mathbb{Z}[x]$ be the ideal generated by* 13 *and* $x - 4$*. Find an integer m such that* $0 \leqslant m \leqslant 12$ *and*

$$(x^{26} + x + 1)^{73} - m \in \mathfrak{I} \ .$$

Problem 18 *Prove that any finite group is isomorphic to*

1. *a subgroup of the group of permutations of n objects*
2. *a subgroup of the group of permutations of n objects which consists only of even permutations.*