Preliminary Exam - Fall 1995

Problem 1 *Let G be a group generated by n elements. Find an upper bound $N(n,k)$ for the number of subgroups H of G with the index $[G:H]=k$.*

Problem 2 *Let A be a finite subset of the unit disc in the plane, and let $N(A,r)$ be the set of points at distance $\leqslant r$ from A, where $0<r<1$. Show that the length of the boundary $N(A,r)$ is, at most, C/r for some constant C independent of A.*

Problem 3 *Find the radius of convergence R of the Taylor series about $z=1$ of the function $f(z)=1/(1+z^2+z^4+z^6+z^8+z^{10})$. Express the answer in terms of real numbers and square roots only.*

Problem 4 *Suppose A and B are real $n\times n$ matrices and C is a complex $n\times n$ matrix such that*

$$CAC^{-1}=B\ .$$

Find a real $n\times n$ matrix D such that $DAD^{-1}=B$.

Problem 5 *Prove that $\mathbb{Q}\,[x,y]/\langle x^2+y^2-1\rangle$ is an integral domain and that its field of fractions is isomorphic to the field of rational functions $\mathbb{Q}\,(t)$.*

Problem 6 *Determine all real numbers $L>1$ so that the boundary value problem*

$$x^2y''(x)+y(x)=0, \qquad 1\leqslant x\leqslant L$$
$$y(1)=y(L)=0$$

has a nonzero solution.

Problem 7 *Let $g(z)=\sum_{n=0}^{\infty} g_n z^n$ and $h(z)=\sum_{n=0}^{\infty} h_n z^n$ be entire functions. Find a formula for the coefficients f_n in the Taylor expansion about $z=0$ of*

$$f(z)=\frac{1}{2\pi i}\int_{|w|=1} g(z/w)h(w)\ \frac{dw}{w}\ \cdot$$

Problem 8 *Show that an* $n \times n$ *matrix of complex numbers* A *satisfying*

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

for $1 \leqslant i \leqslant n$ *must be invertible.*

Problem 9 *Let* x_1 *be a real number,* $0 < x_1 < 1$*, and define a sequence by* $x_{n+1} = x_n - x_n^{n+1}$*. Show that* $\liminf_{n \to \infty} x_n > 0$.

Problem 10 *Let* $\mathbf{F}$ *be a field and* $\mathbf{F}^*$ *be the multiplicative group of nonzero elements. Let* G *be a subgroup of* $\mathbf{F}^*$ *of finite order* n*. Show that* G *is cyclic.*

Problem 11 *Let* $f(z) = u(z) + iv(z)$ *be holomorphic on* $|z| < 1$*,* u *and* v *real. Show that*

$$\int_0^{2\pi} u(re^{i\theta})^2 d\theta = \int_0^{2\pi} v(re^{i\theta})^2 d\theta$$

for $0 < r < 1$ *if* $u(0)^2 = v(0)^2$.

Problem 12 *Let* f *and* f' *be continuous on* $[0, \infty)$ *and* $f(x) = 0$ *for* $x \geqslant 10^{10}$*. Show that*

$$\int_0^{\infty} f(x)^2 dx \leqslant 2 \sqrt{\int_0^{\infty} x^2 f(x)^2 dx} \sqrt{\int_0^{\infty} f'(x)^2 dx} \quad .$$

Problem 13 *Show that*

$$(1+z+z^2+\cdots+z^9)(1+z^{10}+z^{20}+\cdots+z^{90})(1+z^{100}+z^{200}+\cdots+z^{900})\cdots = \frac{1}{1-z}$$

for $|z| < 1$.

Problem 14 *Let* $f(x) \in \mathbb{Q}[x]$ *be a polynomial with rationalcoefficients. Show that there is a* $g(x) \in \mathbb{Q}[x]$*,* $g \neq 0$*, such that* $f(x)g(x) = a_2x^2 + a_3x^3 + a_5x^5 + \cdots + a_px^p$ *is a polynomial in which only prime exponents appear.*

Problem 15 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a* C^∞ *function. Assume that* $f(x)$ *has a local minimum at* $x = 0$*. Prove there is a disc centered on the* y *axis which lies above the graph of* f *and touches the graph at* $(0, f(0))$.

Problem 16 *Let A and B be nonsimilar $n \times n$ complex matrices with the same minimal and the same characteristic polynomial. Show that $n \geqslant 4$ and the minimal polynomial is not equal to the characteristic polynomial.*

Problem 17 *Let $f_1, f_2, \ldots, f_n$ be continuous real valued functions on $[a, b]$. Show that the set $\{f_1, \ldots, f_n\}$ is linearly dependent on $[a, b]$ if and only if*

$$\det\left(\int_a^b f_i(x) f_j(x) dx\right) = 0 \; .$$

Problem 18 *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero C^∞ function such that $f(x)f(y) = f\left(\sqrt{x^2 + y^2}\right)$ for all x and y and that $f(x) \to 0$ as $|x| \to \infty$.*

1. *Prove that f is an even function and that $f(0)$ is 1.*

2. *Prove that f satisfies the differential equation $f'(x) = f''(0)xf(x)$, and find the most general function satisfying the given conditions.*