Preliminary Exam - Fall 1994

Problem 1 *For which values of the real number a does the series*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\sin\frac{1}{n}\right)^{a}$$

converge?

Problem 2 *Prove that the matrix*

$$\begin{pmatrix} 1 & 1.00001 & 1 \\ 1.00001 & 1 & 1.00001 \\ 1 & 1.00001 & 1 \end{pmatrix}$$

has one positive eigenvalue and one negative eigenvalue.

Problem 3 *Evaluate the integrals*

$$\int_{-\pi}^{\pi}\frac{\sin n\theta}{\sin\theta}d\theta, \quad n=1,2,\ldots.$$

Problem 4 *Suppose the group G has a nontrivial subgroup H which is contained in every nontrivial subgroup of G. Prove that H is contained in the center of G.*

Problem 5

1. *Find a basis for the space of real solutions of the differential equation*

$$(*) \qquad \sum_{n=0}^{7}\frac{d^n x}{dt^n}=0.$$

2. *Find a basis for the subspace of real solutions of $(*)$ that satisfy*

$$\lim_{t\to+\infty}x(t)=0.$$

Problem 6 *Let* $A = (a_{ij})_{i,j=1}^n$ *be a real* $n \times n$ *matrix such that* $a_{ii} \geqslant 1$ *for all* i*, and*

$$\sum_{i \neq j} a_{ij}^2 < 1.$$

Prove that A *is invertible.*

Problem 7 *Let* f *be a continuously differentiable function from* $\mathbb{R}^2$ *into* $\mathbb{R}$*. Prove that there is a continuous one-to-one function* g *from* $[0,1]$ *into* $\mathbb{R}^2$ *such that the composite function* $f \circ g$ *is constant.*

Problem 8 *Let* $\mathbb{Q}$ *be the field of rational numbers. For* θ *a real number, let* $\mathbf{F}_\theta = \mathbb{Q}(\sin\theta)$ *and* $\mathbf{E}_\theta = \mathbb{Q}\left(\sin\frac{\theta}{3}\right)$*. Show that* $\mathbf{E}_\theta$ *is an extension field of* $\mathbf{F}_\theta$*, and determine all possibilities for* $\dim_{\mathbf{F}_\theta}\mathbf{E}_\theta$*.*

Problem 9 *Evaluate*

$$\int_0^\infty \frac{(\log x)^2}{x^2+1}\,dx\,.$$

Problem 10 *Let the function* $f : \mathbb{R}^n \to \mathbb{R}^n$ *satisfy the following two conditions:*

(i) $f(K)$ *is compact whenever* K *is a compact subset of* $\mathbb{R}^n$*.*

(ii) If $\{K_n\}$ *is a decreasing sequence of compact subsets of* $\mathbb{R}^n$*, then*

$$f\left(\bigcap_1^\infty K_n\right) = \bigcap_1^\infty f(K_n)\,.$$

Prove that f *is continuous.*

Problem 11 *Write down a list of 5×5 complex matrices, as long as possible, with the following properties:*

1. *The characteristic polynomial of each matrix in the list is* x^5*;*
2. *The minimal polynomial of each matrix in the list is* x^3*;*
3. *No two matrices in the list are similar.*

Problem 12 *Suppose the coefficients of the power series*

$$\sum_{n=0}^{\infty} a_n z^n$$

are given by the recurrence relation

$$a_0 = 1,\ a_1 = -1,\ 3a_n + 4a_{n-1} - a_{n-2} = 0, \quad n = 2, 3, \ldots.$$

Find the radius of convergence of the series and the function to which it converges in its disc of convergence.

Problem 13 *Let p be an odd prime and $\mathbf{F}_p$ the field of p elements. How many elements of $\mathbf{F}_p$ have square roots in $\mathbf{F}_p$? How many have cube roots in $\mathbf{F}_p$?*

Problem 14 *Find the maximum area of all triangles that can be inscribed in an ellipse with semiaxes a and b, and describe the triangles that have maximum area.*
Note: See also Problem **??**.

Problem 15 *Let $M_{7\times 7}$ denote the vector space of real 7×7 matrices. Let A be a diagonal matrix in $M_{7\times 7}$ that has $+1$ in four diagonal positions and -1 in three diagonal positions. Define the linear transformation T on $M_{7\times 7}$ by $T(X) = AX - XA$. What is the dimension of the range of T?*

Problem 16 *Let $\mathcal{D}$ denote the open unit disc in $\mathbb{R}^2$. Let u be an* eigenfunction for the Laplacian *in $\mathcal{D}$; that is, a real valued function of class C^2 defined in $\overline{\mathcal{D}}$, zero on the boundary of $\mathcal{D}$ but not identically zero, and satisfying the differential equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \lambda u\,,$$

where λ is a constant. Prove that

$$(*) \qquad \iint_{\mathcal{D}} |\mathrm{grad}\, u|^2\, dxdy + \lambda \iint_{\mathcal{D}} u^2 dxdy = 0\,,$$

and hence that $\lambda < 0$.

Problem 17 *Let R be a ring with identity, and let u be an element of R with a right inverse. Prove that the following conditions on u are equivalent:*

1. *u has more than one right inverse;*
2. *u is a zero divisor;*
3. *u is not a unit.*

Problem 18 *Let the function f be analytic in the complex plane, real on the real axis, 0 at the origin, and not identically 0. Prove that if f maps the imaginary axis into a straight line, then that straight line must be either the real axis or the imaginary axis.*