Problem 1 For which values of the real number $a$ does the series
\[ \sum_{n=1}^{\infty} \left( \frac{1}{n} - \sin \frac{1}{n} \right)^a \]
converge?

Problem 2 Prove that the matrix
\[
\begin{pmatrix}
1 & 1.00001 & 1 \\
1.00001 & 1 & 1.00001 \\
1 & 1.00001 & 1
\end{pmatrix}
\]
has one positive eigenvalue and one negative eigenvalue.

Problem 3 Evaluate the integrals
\[ \int_{-\pi}^{\pi} \frac{\sin n\theta}{\sin \theta} d\theta, \quad n = 1, 2, \ldots. \]

Problem 4 Suppose the group $G$ has a nontrivial subgroup $H$ which is contained in every nontrivial subgroup of $G$. Prove that $H$ is contained in the center of $G$.

Problem 5 1. Find a basis for the space of real solutions of the differential equation
\[ (*) \quad \sum_{n=0}^{7} \frac{d^n x}{dt^n} = 0. \]

2. Find a basis for the subspace of real solutions of $(*)$ that satisfy
\[ \lim_{t \to +\infty} x(t) = 0. \]
Problem 6  Let $A = (a_{ij})_{i,j=1}^n$ be a real $n \times n$ matrix such that $a_{ii} \geq 1$ for all $i$, and
\[ \sum_{i \neq j} a_{ij}^2 < 1. \]
Prove that $A$ is invertible.

Problem 7  Let $f$ be a continuously differentiable function from $\mathbb{R}^2$ into $\mathbb{R}$. Prove that there is a continuous one-to-one function $g$ from $[0,1]$ into $\mathbb{R}^2$ such that the composite function $f \circ g$ is constant.

Problem 8  Let $\mathbb{Q}$ be the field of rational numbers. For $\theta$ a real number, let $F_\theta = \mathbb{Q}(\sin \theta)$ and $E_\theta = \mathbb{Q}(\sin \sqrt{\theta})$. Show that $E_\theta$ is an extension field of $F_\theta$, and determine all possibilities for $\dim_{F_\theta} E_\theta$.

Problem 9  Evaluate
\[ \int_0^\infty \frac{(\log x)^2}{x^2 + 1} \, dx. \]

Problem 10  Let the function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following two conditions:

(i) $f(K)$ is compact whenever $K$ is a compact subset of $\mathbb{R}^n$.

(ii) If $\{K_n\}$ is a decreasing sequence of compact subsets of $\mathbb{R}^n$, then
\[ f \left( \bigcap_{n=1}^\infty K_n \right) = \bigcap_{n=1}^\infty f(K_n). \]
Prove that $f$ is continuous.

Problem 11  Write down a list of $5 \times 5$ complex matrices, as long as possible, with the following properties:

1. The characteristic polynomial of each matrix in the list is $x^5$;
2. The minimal polynomial of each matrix in the list is $x^3$;
3. No two matrices in the list are similar.
Problem 12 Suppose the coefficients of the power series
\[ \sum_{n=0}^{\infty} a_n z^n \]
are given by the recurrence relation
\[ a_0 = 1, \ a_1 = -1, \ 3a_n + 4a_{n-1} - a_{n-2} = 0, \ n = 2, 3, \ldots. \]
Find the radius of convergence of the series and the function to which it converges in its disc of convergence.

Problem 13 Let \( p \) be an odd prime and \( \mathbb{F}_p \) the field of \( p \) elements. How many elements of \( \mathbb{F}_p \) have square roots in \( \mathbb{F}_p \)? How many have cube roots in \( \mathbb{F}_p \)?

Problem 14 Find the maximum area of all triangles that can be inscribed in an ellipse with semiaxes \( a \) and \( b \), and describe the triangles that have maximum area.
Note: See also Problem \( ?? \).

Problem 15 Let \( M_{7 \times 7} \) denote the vector space of real \( 7 \times 7 \) matrices. Let \( A \) be a diagonal matrix in \( M_{7 \times 7} \) that has \(+1\) in four diagonal positions and \(-1\) in three diagonal positions. Define the linear transformation \( T \) on \( M_{7 \times 7} \) by \( T(X) = AX -XA \). What is the dimension of the range of \( T \)?

Problem 16 Let \( \mathcal{D} \) denote the open unit disc in \( \mathbb{R}^2 \). Let \( u \) be an eigenfunction for the Laplacian in \( \mathcal{D} \); that is, a real valued function of class \( C^2 \) defined in \( \mathcal{D} \), zero on the boundary of \( \mathcal{D} \) but not identically zero, and satisfying the differential equation
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \lambda u, \]
where \( \lambda \) is a constant. Prove that
\[ (\ast) \quad \int \int_{\mathcal{D}} |\text{grad } u|^2 dxdy + \lambda \int \int_{\mathcal{D}} u^2 dxdy = 0, \]
and hence that \( \lambda < 0 \).

Problem 17 Let \( R \) be a ring with identity, and let \( u \) be an element of \( R \) with a right inverse. Prove that the following conditions on \( u \) are equivalent:
1. \( u \) has more than one right inverse;

2. \( u \) is a zero divisor;

3. \( u \) is not a unit.

Problem 18 Let the function \( f \) be analytic in the complex plane, real on the real axis, 0 at the origin, and not identically 0. Prove that if \( f \) maps the imaginary axis into a straight line, then that straight line must be either the real axis or the imaginary axis.