Preliminary Exam - Fall 1993

Problem 1 *Let X be a metric space and (x_n) a convergent sequence in X with limit x_0. Prove that the set $C = \{x_0, x_1, x_2, ...\}$ is compact.*

Problem 2 *Let A be the additive group of rational numbers, and let M be the multiplicative group of positive rational numbers. Determine all homomorphisms of A into M.*

Problem 3 *Describe the region in the complex plane where the infinite series*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(\frac{nz}{z-2}\right)$$

converges. Draw a sketch of the region.

Problem 4 *Let $\mathbf{F}$ be a field. For m and n positive integers, let $M_{m\times n}$ be the vector space of $m \times n$ matrices over $\mathbf{F}$. Fix m and n, and fix matrices A and B in $M_{m\times n}$. Define the linear transformation T from $M_{n\times m}$ to $M_{m\times n}$ by*

$$T(X) = AXB.$$

Prove that if $m \neq n$, then T is not invertible.

Problem 5 *Let the function $x(t)$ $(-\infty < t < \infty)$ be a solution of the differential equation*

$$\frac{d^2x}{dt^2} - 2b\,\frac{dx}{dt} + cx = 0$$

such that $x(0) = x(1) = 0$. (Here, b and c are real constants.) Prove that$x(n) = 0$ for every integer n.

Problem 6 *Let A, B, and C be finite abelian groups. Prove that if $A \times B$ is isomorphic to $A \times C$, then B is isomorphic to C.*

Problem 7 *Let $M_{n\times n}$ $(n \geqslant 2)$ be the space of real $n\times n$ matrices, identified in the usual way with the Euclidean space $\mathbb{R}^{n^2}$. Let F be the determinant map of $M_{n\times n}$ into $\mathbb{R}$: $F(X) = \det(X)$. Find all of the* critical points *of F; that is, all matrices X such that $DF(X) = 0$.*

Problem 8 *Prove that if* A *is an* $n \times n$ *matrix over* $\mathbb{C}$, *and if* $A^k = I$ *for some positive integer* k, *then* A *is diagonalizable.*

Problem 9 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2 - 2x + 4}\, dx\,.$$

Problem 10 *Let* f *be a continuous real valued function on* $[0, \infty)$. *Let* A *be the set of real numbers* a *that can be expressed as* $a = \lim_{n\to\infty} f(x_n)$ *for some sequence* (x_n) *in* $[0, \infty)$ *such that* $\lim_{n\to\infty} x_n = \infty$. *Prove that if* A *contains the two numbers* a *and* b, *then contains the entire interval with endpoints* a *and* b.

Problem 11 *Let* R *be a commutative ring with identity. Let* G *be a finite subgroup of* R^*, *the group of units of* R. *Prove that if* R *is an integral domain, then* G *is cyclic.*

Problem 12 *Evaluate the integral* $\frac{1}{2\pi i}\int_\gamma f(z)\ dz$ *for the function*$f(z) = z^{-2}(1 - z^2)^{-1}e^z$ *and the curve* γ *depicted by*

`../Fig/Pr/Fa93-12.ps` *not found*

Problem 13 *Show that there are at least two nonisomorphic nonabelian groups of order* 40.

Problem 14 *Let* n *be an integer larger than* 1. *Is there a differentiable function on* $[0, \infty)$ *whose derivative equals its* n^{th} *power and whose value at the origin is positive?*

Problem 15 *Prove that the matrix*

$$\begin{pmatrix} 0 & 5 & 1 & 0 \\ 5 & 0 & 5 & 1 \\ 1 & 5 & 0 & 5 \\ 0 & 1 & 5 & 0 \end{pmatrix}$$

has two positive and two negative eigenvalues (taking into account multiplicities).

Problem 16 *Let K be a continuous real valued function defined on $[0,1] \times [0,1]$. Let F be the family of functions f on $[0,1]$ of the form*

$$f(x) = \int_0^1 g(y)K(x,y)\,dy$$

with g a real valued continuous function on $[0,1]$ satisfying $|g| \leqslant 1$ everywhere. Prove that the family F is equicontinuous.

Problem 17 *Let w be a positive continuous function on $[0,1]$, n a positive integer, and P_n the vector space of real polynomials whose degrees are at most n, equipped with the inner product*

$$\langle p, q \rangle = \int_0^1 p(t)q(t)w(t)\ dt\,.$$

1. *Prove that P_n has an orthonormal basis $p_0, p_1, \ldots, p_n$ (i.e., $\langle p_j, p_k \rangle = 1$ for $j = k$ and 0 for $j \neq k$) such that $\deg p_k = k$ for each k.*
2. *Prove that $\langle p_k, p_k' \rangle = 0$ for each k.*

Problem 18 *Let f be a continuous real valued function on $[0,1]$, and let the function h in the complex plane be defined by*

$$h(z) = \int_0^1 f(t)\cos zt\,dt.$$

1. *Prove that h is analytic in the entire plane.*
2. *Prove that h is the zero function only if f is the zero function.*