Preliminary Exam - Fall 1992

Problem 1 *Are the matrices* $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix}$ *and* $\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ *similar?*

Problem 2 *Let* $\mathfrak{I}$ *denote the ideal in* $\mathbb{Z}[x]$, *the ring of polynomials with coefficients in* $\mathbb{Z}$, *generated by* $x^3 + x + 1$ *and* 5. *Is* $\mathfrak{I}$ *a prime ideal?*

Problem 3

1. *How many roots does the polynomial defined by* $p(z) = 2z^5 + 4z^2 + 1$ *have in the disc* $|z| < 1$*?*
2. *How many roots does the same polynomial have on the real axis?*

Problem 4 *Let the real valued function* $y(t)$ $(0 \leqslant t < \infty)$ *solve the initial value problem*

$$y'' = -|y|, \quad y(0) = 1,\ y'(0) = 0.$$

Prove that there is exactly one $t > 0$ *such that* $y(t) = 0$.

Problem 5 *Let* G *be a group and* H *and* K *subgroups such that* H *has a finite index in* G. *Prove that* $K \cap H$ *has a finite index in* K.

Problem 6 *Let* (X_1, d_1) *and* (X_2, d_2) *be metric spaces and*$f : X_1 \to X_2$ *a continuous surjective map such that* $d_1(p, q) \leqslant d_2(f(p), f(q))$ *for every pair of points* p, q *in* X_1.

1. *If* X_1 *is complete, must* X_2 *be complete? Give a proof or a counterexample.*
2. *If* X_2 *is complete, must* X_1 *be complete? Give a proof or a counterexample.*

Problem 7 *Evaluate*

$$\int_C \frac{e^z}{z(2z+1)^2}\, dz,$$

where C *is the unit circle with counterclockwise orientation.*

Problem 8 *Let* $\mathbf{F}$ *be a field,* V *a finite-dimensional vector space over* $\mathbf{F}$*, and* T *a linear transformation of* V *into* V *whose minimum polynomial,* μ*, is irreducible over* $\mathbf{F}$*.*

1. *Let* v *be a nonzero vector in* V *and let* V_1 *be the subspace spanned by* v *and its images under the positive powers of* T*. Prove that* $\dim V_1 = \deg \mu$*.*

2. *Prove that* $\deg \mu$ *divides* $\dim V$*.*

Problem 9 *Let the function* f *be analytic in the region* $|z| > 1$ *of the complex plane. Prove that if* f *is real valued on the interval* $(1, \infty)$ *of the real axis, then* f *is also real valued on the interval* $(-\infty, -1)$*.*

Problem 10 *How many Sylow* 2*-subgroups does the dihedral group* D_n *of order* $2n$ *have, when* n *is odd?*

Problem 11 *Let* V *be a finite-dimensional vector space over a field* $\mathbf{F}$*, and let* $B : V \times V \to \mathbf{F}$ *be a bilinear map (not necessarily symmetric). Define the subspaces* V_1 *and* V_2 *by*

$$V_1 = \{x \in V : B(x, y) = 0 \quad \textit{for all} \quad y \quad \textit{in} \quad V\}$$

$$V_2 = \{y \in V : B(x, y) = 0 \quad \textit{for all} \quad x \quad \textit{in} \quad V\}.$$

Prove that $\dim V_1 = \dim V_2$*.*

Problem 12 *Let* $\{f_n\}$ *be a sequence of real valued* C^1 *functions on* $[0, 1]$ *such that, for all* n*,*

$$|f_n'(x)| \leqslant \frac{1}{\sqrt{x}} \quad (0 < x \leqslant 1),$$

$$\int_0^1 f_n(x)\, dx = 0\,.$$

Prove that the sequence has a subsequence that converges uniformly on $[0, 1]$*.*

Problem 13 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(1 + x^2)}\, dx\,.$$

Problem 14 *Let f and g be positive continuous functions on $\mathbb{R}$, with $g \leqslant f$ everywhere. Assume the initial value problem*

$$\frac{dx}{dt} = f(x), \quad x(0) = 0,$$

has a solution defined on all of $\mathbb{R}$. Prove that the initial value problem

$$\frac{dx}{dt} = g(x), \quad x(0) = 0,$$

also has a solution defined on all of $\mathbb{R}$.

Problem 15 *Let G be the group of all real 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ with $a > 0$. Let N be the subgroup of those matrices in G having $a = 1$.*

(a) Prove that N is a normal subgroup of G and that G/N is isomorphic to $\mathbb{R}$.

(b) Find a proper normal subgroup of G that contains N properly.

Problem 16 *Let f be a C^1 function of $\mathbb{R}^n$ into $\mathbb{R}^n$ such that Df has rank n everywhere. Assume f is proper (which, by definition, means $f^{-1}(K)$ is compact whenever K is compact). Prove that $f(\mathbb{R}^n) = \mathbb{R}^n$.*

Problem 17 *Let s be a real number, and let the function u be defined in $\mathbb{C}\backslash(-\infty, 0]$ by*

$$u(re^{i\theta}) = r^s \cos s\theta \qquad (r > 0, \quad -\pi < \theta < \pi).$$

Prove that u is a harmonic function.

Problem 18 *Let k be a positive integer. Determine those real numbers c for which every sequence (x_n) of real numbers satisfying the recurrence relation*

$$\frac{1}{2}(x_{n+1} + x_{n-1}) = cx_n$$

has period k (i.e., $x_{n+k} = x_n$ for all n).