Preliminary Exam - Fall 1991

Problem 1 *Prove that every finite group of order at least* 3 *has a nontrivial automorphism.*

Problem 2 *Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that $|f(x) - f(y)| \geqslant |x - y|$ for all x and y. Prove that the range of f is all of $\mathbb{R}$.*
Note: See also Problem **??**.

Problem 3 *Evaluate the integral*

$$I = \frac{1}{2\pi i} \int_C \frac{z^{n-1}}{3z^n - 1}\, dz,$$

where n is a positive integer, and C is the circle $|z| = 1$, with counterclockwise orientation.

Problem 4

1. *Prove that any real $n \times n$ matrix M can be written as $M = A + S + cI$, where A is antisymmetric, S is symmetric, c is a scalar, I is the identity matrix, and $\operatorname{tr} S = 0$.*

2. *Prove that with the above notation,*

$$\operatorname{tr}(M^2) = \operatorname{tr}(A^2) + \operatorname{tr}(S^2) + \frac{1}{n}(\operatorname{tr} M)^2.$$

Problem 5 *Let f be an infinitely differentiable function from $\mathbb{R}$ to $\mathbb{R}$. Suppose that, for some positive integer n,*

$$f(1) = f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 0.$$

Prove that $f^{(n+1)}(x) = 0$ for some x in $(0, 1)$.

Problem 6 *Let the function f be analytic in the disc $|z| < 1$ of the complex plane. Assume that there is a positive constant M such that*

$$\int_0^{2\pi} |f'(re^{i\theta})|\, d\theta \leqslant M, \qquad (0 \leqslant r < 1).$$

Prove that

$$\int_{[0,1)} |f(x)|\, dx < \infty.$$

Problem 7 *Consider the vector differential equation*

$$\frac{dx(t)}{dt} = A(t)x(t)$$

where A is a smooth $n \times n$ function on $\mathbb{R}$. Assume A has the property that $\langle \mathrm{A}(t)y, y\rangle \leqslant c\|y\|^2$ for all y in $\mathbb{R}^n$ and all t, where c is a fixed real number. Prove that any solution $x(t)$ of the equation satisfies $\|x(t)\| \leqslant e^{ct}\|x(0)\|$ for all $t > 0$.

Problem 8 *Let $a_1, a_2, a_3, \ldots$ be positive numbers.*

1. *Prove that $\sum a_n < \infty$ implies $\sum \sqrt{a_n a_{n+1}} < \infty$.*
2. *Prove that the converse of the above statement is false.*

Problem 9 *Let G be a group of order $2p$, where p is an odd prime. Assume that G has a normal subgroup of order 2. Prove that G is cyclic.*

Problem 10 *Let $\mathbf{F}$ be a finite field of order p. Compute the order of $SL_3(\mathbf{F})$, the group of 3×3 matrices over $\mathbf{F}$ of determinant 1.*

Problem 11 *Let X and Y be metric spaces and f a continuous map of X into Y. Let $K_1, K_2, \ldots$ be nonempty compact subsets of X such that$K_{n+1} \subset K_n$ for all n, and let $K = \bigcap K_n$. Prove that $f(K) = \bigcap f(K_n)$.*

Problem 12 *Let p be a nonconstant complex polynomial whose zeros are all in the half-plane $\Im z > 0$.*

1. *Prove that $\Im(p'/p) > 0$ on the real axis.*
2. *Find a relation between $\deg p$ and*

$$\int_{-\infty}^{\infty} \Im \frac{p'(x)}{p(x)}\, dx\,.$$

Problem 13 *Let $A = (a_{ij})_{i,j=1}^n$ be a real $n \times n$ matrix with nonnegative entries such that*

$$\sum_{j=1}^n a_{ij} = 1 \qquad (1 \leqslant i \leqslant n).$$

Prove that no eigenvalue of A has absolute value greater than 1.

Problem 14 *Let $\mathcal{B}$ denote the unit ball of $\mathbb{R}^3$, $\mathcal{B} = \{r \in \mathbb{R}^3 \mid \|r\| \leqslant 1\}$. Let $J = (J_1, J_2, J_3)$ be a smooth vector field on $\mathbb{R}^3$ that vanishes outside of $\mathcal{B}$ and satisfies $\nabla \cdot \vec{J} = 0$.*

1. *For f a smooth, scalar-valued function defined on a neighborhood of $\mathcal{B}$, prove that*
$$\int_{\mathcal{B}} (\nabla f) \cdot \vec{J} \, dxdydz = 0.$$
2. *Prove that*
$$\int_{\mathcal{B}} J_1 \, dxdydz = 0.$$

Problem 15 *Let $\mathfrak{I}$ be the ideal in the ring $\mathbb{Z}[x]$ generated by $x - 7$ and 15. Prove that the quotient ring $\mathbb{Z}[x]/\mathfrak{I}$ is isomorphic to $\mathbb{Z}_{15}$.*

Problem 16 *Let $M_{n\times n}$ be the space of real $n\times n$ matrices. Regard it as a metric space with the distance function*

$$d(A, B) = \sum_{i,j=1}^{n} |a_{ij} - b_{ij}| \qquad (A = (a_{ij}), B = (b_{ij}))\,.$$

Prove that the set of nilpotent matrices in $M_{n\times n}$ is a closed set.

Problem 17 *Let f be a C^1 function from the interval $(-1, 1)$ into $\mathbb{R}^2$ such that $f(0) = 0$ and $f'(0) \neq 0$. Prove that there is a number ε in $(0, 1)$ such that $\|f(t)\|$ is an increasing function of t on $(0, \varepsilon)$.*

Problem 18 *Let the function f be analytic in the entire complex plane and satisfy the inequality $|f(z)| \leqslant |\Re z|^{-1/2}$ off the imaginary axis. Prove that f is constant.*