Problem 1 Prove that every finite group of order at least 3 has a nontrivial automorphism.

Problem 2 Let $f$ be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that $|f(x) - f(y)| \geq |x - y|$ for all $x$ and $y$. Prove that the range of $f$ is all of $\mathbb{R}$.
Note: See also Problem ??.

Problem 3 Evaluate the integral
\[ I = \frac{1}{2\pi i} \int_C \frac{z^{n-1}}{3z^n - 1} dz, \]
where $n$ is a positive integer, and $C$ is the circle $|z| = 1$, with counterclockwise orientation.

Problem 4 1. Prove that any real $n \times n$ matrix $M$ can be written as $M = A + S + cI$, where $A$ is antisymmetric, $S$ is symmetric, $c$ is a scalar, $I$ is the identity matrix, and $\text{tr } S = 0$.

2. Prove that with the above notation,
\[ \text{tr}(M^2) = \text{tr}(A^2) + \text{tr}(S^2) + \frac{1}{n}(\text{tr } M)^2. \]

Problem 5 Let $f$ be an infinitely differentiable function from $\mathbb{R}$ to $\mathbb{R}$. Suppose that, for some positive integer $n$,
\[ f(1) = f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 0. \]
Prove that $f^{(n+1)}(x) = 0$ for some $x$ in $(0,1)$.

Problem 6 Let the function $f$ be analytic in the disc $|z| < 1$ of the complex plane. Assume that there is a positive constant $M$ such that
\[ \int_0^{2\pi} |f'(re^{i\theta})| d\theta \leq M, \quad (0 \leq r < 1). \]
Prove that
\[ \int_{[0,1]} |f(x)| dx < \infty. \]
Problem 7 Consider the vector differential equation
\[
\frac{dx(t)}{dt} = A(t)x(t)
\]
where \( A \) is a smooth \( n \times n \) function on \( \mathbb{R} \). Assume \( A \) has the property that
\[
\langle A(t)y, y \rangle \leq c\|y\|^2
\]
for all \( y \) in \( \mathbb{R}^n \) and all \( t \), where \( c \) is a fixed real number. Prove that any solution \( x(t) \) of the equation satisfies \( \|x(t)\| \leq e^{ct}\|x(0)\| \) for all \( t > 0 \).

Problem 8 Let \( a_1, a_2, a_3, \ldots \) be positive numbers.

1. Prove that \( \sum a_n < \infty \) implies \( \sum \sqrt{a_n a_{n+1}} < \infty \).

2. Prove that the converse of the above statement is false.

Problem 9 Let \( G \) be a group of order \( 2p \), where \( p \) is an odd prime. Assume that \( G \) has a normal subgroup of order \( 2 \). Prove that \( G \) is cyclic.

Problem 10 Let \( F \) be a finite field of order \( p \). Compute the order of \( SL_3(F) \), the group of \( 3 \times 3 \) matrices over \( F \) of determinant 1.

Problem 11 Let \( X \) and \( Y \) be metric spaces and \( f \) a continuous map of \( X \) into \( Y \). Let \( K_1, K_2, \ldots \) be nonempty compact subsets of \( X \) such that \( K_{n+1} \subset K_n \) for all \( n \), and let \( K = \bigcap K_n \). Prove that \( f(K) = \bigcap f(K_n) \).

Problem 12 Let \( p \) be a nonconstant complex polynomial whose zeros are all in the half-plane \( \Im z > 0 \).

1. Prove that \( \Im(p'/p) > 0 \) on the real axis.

2. Find a relation between \( \deg p \) and
\[
\int_{-\infty}^{\infty} \Im \frac{p'(x)}{p(x)} \, dx.
\]

Problem 13 Let \( A = (a_{ij})_{i,j=1}^n \) be a real \( n \times n \) matrix with nonnegative entries such that
\[
\sum_{j=1}^n a_{ij} = 1 \quad (1 \leq i \leq n).
\]
Prove that no eigenvalue of \( A \) has absolute value greater than 1.
**Problem 14** Let $\mathcal{B}$ denote the unit ball of $\mathbb{R}^3$, $\mathcal{B} = \{ r \in \mathbb{R}^3 \mid \| r \| \leq 1 \}$. Let $J = (J_1, J_2, J_3)$ be a smooth vector field on $\mathbb{R}^3$ that vanishes outside of $\mathcal{B}$ and satisfies $\nabla \cdot J = 0$.

1. For $f$ a smooth, scalar-valued function defined on a neighborhood of $\mathcal{B}$, prove that
$$\int_{\mathcal{B}} (\nabla f) \cdot \vec{J} \, dx dy dz = 0.$$ 

2. Prove that
$$\int_{\mathcal{B}} J_1 \, dx dy dz = 0.$$

**Problem 15** Let $\mathcal{I}$ be the ideal in the ring $\mathbb{Z}[x]$ generated by $x - 7$ and $15$. Prove that the quotient ring $\mathbb{Z}[x]/\mathcal{I}$ is isomorphic to $\mathbb{Z}_{15}$.

**Problem 16** Let $M_{n \times n}$ be the space of real $n \times n$ matrices. Regard it as a metric space with the distance function
$$d(A, B) = \sum_{i,j=1}^{n} |a_{ij} - b_{ij}| \quad (A = (a_{ij}), B = (b_{ij})).$$

Prove that the set of nilpotent matrices in $M_{n \times n}$ is a closed set.

**Problem 17** Let $f$ be a $C^1$ function from the interval $(-1, 1)$ into $\mathbb{R}^2$ such that $f(0) = 0$ and $f'(0) \neq 0$. Prove that there is a number $\varepsilon$ in $(0, 1)$ such that $\| f(t) \|$ is an increasing function of $t$ on $(0, \varepsilon)$.

**Problem 18** Let the function $f$ be analytic in the entire complex plane and satisfy the inequality $|f(z)| \leq |\Re z|^{-1/2}$ off the imaginary axis. Prove that $f$ is constant.