Preliminary Exam - Fall 1990

Problem 1 *Find all pairs of integers* a *and* b *satisfying* $0 < a < b$ *and* $a^b = b^a$.

Problem 2 *Evaluate the integral*

$$I = \frac{1}{2\pi i} \int_C \frac{dz}{(z-2)(1+2z)^2(1-3z)^3}$$

where C *is the circle* $|z| = 1$ *with counterclockwise orientation.*

Problem 3 *Let* R *be a ring with identity, and let* $\mathfrak{I}$ *be the left ideal of* R *generated by* $\{ab - ba \mid a, b \in R\}$. *Prove that* $\mathfrak{I}$ *is a two-sided ideal.*

Problem 4 *Suppose* f *is a continuous real valued function. Show that*

$$\int_0^1 f(x)x^2\,dx = \frac{1}{3}f(\xi)$$

for some $\xi \in [0, 1]$.

Problem 5 *Let* A *be a real symmetric* $n \times n$ *matrix that is positive definite. Let* $y \in \mathbb{R}^n$, $y \neq 0$. *Prove that the limit*

$$\lim_{m\to\infty} \frac{y^t A^{m+1} y}{y^t A^m y}$$

exists and is an eigenvalue of A.

Problem 6 *Let the function* f *be analytic in the entire complex plane, and suppose that* $f(z)/z \to 0$ *as* $|z| \to \infty$. *Prove that* f *is constant.*

Problem 7 *Let* G *be a group and* N *be a normal subgroup of* G *with* $N \neq G$. *Suppose that there does not exist a subgroup* H *of* G *satisfying* $N \subset H \subset G$ *and* $N \neq H \neq G$. *Prove that the index of* N *in* G *is finite and equal to a prime number.*

Problem 8 *Let f be a continuous real valued function satisfying $f(x) \geqslant 0$, for all x, and*

$$\int_0^\infty f(x)\,dx < \infty.$$

Prove that

$$\frac{1}{n}\int_0^n xf(x)\,dx \to 0$$

as $n \to \infty$.

Problem 9 *Let $\mathbb{R}^3$ be 3-space with the usual inner product, and$(a, b, c) \in \mathbb{R}^3$ a vector of length 1. Let W be the plane defined by $ax + by + cz = 0$. Find, in the standard basis, the matrix representing the orthogonal projection of $\mathbb{R}^3$ onto W.*

Problem 10 *Determine the Jordan Canonical Form of the matrix*

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}.$$

Problem 11 *Suppose that f maps the compact interval I into itself and that*

$$|f(x) - f(y)| < |x - y|$$

for all $x, y \in I$, $x \neq y$. Can one conclude that there is some constant $M < 1$ such that, for all $x, y \in I$,

$$|f(x) - f(y)| \leqslant M|x - y|\,?$$

Problem 12 *Let A be an additively written abelian group, and$u, v : A \to A$ homomorphisms. Define the group homomorphisms $f, g : A \to A$ by*

$$f(a) = a - v\,(u(a))\,, \quad g(a) = a - u\,(v(a)) \quad (a \in A).$$

Prove that the kernel of f is isomorphic to the kernel of g.

Problem 13 *Suppose that f is analytic on the open upper half-plane and satisfies $|f(z)| \leqslant 1$ for all z, $f(i) = 0$. How large can $|f(2i)|$ be under these conditions?*

Problem 14 *Prove that* $\sqrt{2}+\sqrt[3]{3}$ *is irrational.*

Problem 15 *Let* n *be a positive integer and let* P_{2n+1} *be the vector space of real polynomials whose degrees are, at most,* $2n+1$*. Prove that there exist unique real numbers* $c_1, \ldots, c_n$ *such that, for all* $p \in P_{2n+1}$.

$$\int_{-1}^{1} p(x)\,dx = 2p(0) + \sum_{k=1}^{n} c_k(p(k) + p(-k) - 2p(0))$$

Problem 16 *Evaluate the limit*

$$\lim_{n\to\infty} \cos\frac{\pi}{2^2}\cos\frac{\pi}{2^3}\cdots\cos\frac{\pi}{2^n}.$$

Problem 17 *Does the set* $G = \{a \in \mathbb{R} \mid a > 0, a \neq 1\}$ *form a group with the operation* $a * b = a^{\log b}$*?*

Problem 18 *Let the function* f *be analytic in the entire complex plane and satisfy*

$$\int_{0}^{2\pi} |f(re^{i\theta})|\,d\theta \leqslant r^{17/3}$$

for all $r > 0$*. Prove that* f *is the zero function.*