Preliminary Exam - Fall 1989

Problem 1 *Let A be a finite abelian group, and m the maximum of the orders of the elements of A. Put $S = \{a \in A \mid |a| = m\}$. Prove that A is generated by S.*

Problem 2 *Let $f : [0,1] \to \mathbb{R}$ be a real valued continuously differentiable function with $f(0) = 0$. Suppose also that there is a constant $M > 0$ such that, for $0 \leqslant x \leqslant 1$,*

$$0 \leqslant f'(x) \leqslant M f(x).$$

Prove that $f(x) = 0$ for $0 \leqslant x \leqslant 1$.

Problem 3 *Let A be a real, upper-triangular, $n \times n$ matrix that commutes with its transpose. Prove that A is diagonal.*

Problem 4 *Let $f(z)$ be analytic for $|z| < 1$ and suppose that*

$$|f(z)| \leqslant \frac{1}{1 - |z|}.$$

Show that $|f'(0)| \leqslant 4$.

Problem 5 *Let G be a group, G' its commutator subgroup, and N a normal subgroup of G. Suppose that N is cyclic. Prove that $gn = ng$ for all $g \in G'$ and all $n \in N$.*

Problem 6 *Let $X \subset \mathbb{R}^n$ be a closed set and r a fixed positive real number. Let $Y = \{y \in \mathbb{R}^n \mid |x - y| = r \text{ for some } x \in X\}$. Show that Y is closed.*

Problem 7 *Let A and B be diagonalizable linear transformations of $\mathbb{R}^n$ into itself such that $AB = BA$. Let E be an eigenspace of A. Prove that the restriction of B to E is diagonalizable.*

Problem 8 *Evaluate the integral*

$$I = \int_0^\infty \frac{\log x}{1 + x^2}\, dx\,.$$

Problem 9 *Let* $\mathbf{F}$ *be a field,* $\mathbf{F}[x]$ *the polynomial ring in one variable over* $\mathbf{F}$*, and* R *a subring of* $\mathbf{F}[x]$ *with* $\mathbf{F} \subset R$*. Prove that there exists a finite set* $\{f_1, f_2, \ldots, f_n\}$ *of elements of* $\mathbf{F}[x]$ *such that* $R = \mathbf{F}[f_1, f_2, \ldots, f_n]$*.*

Problem 10 *Let* α *be a number in* $(0, 1)$*. Prove that any sequence* (x_n) *of real numbers satisfying the recurrence relation*

$$x_{n+1} = \alpha x_n + (1 - \alpha)x_{n-1}$$

has a limit, and find an expression for the limit in terms of α*,* x_0 *and* x_1*.*

Problem 11 *Let* φ *be Euler's totient function; so if* n *is a positive integer, then* $\varphi(n)$ *is the number of integers* m *for which* $1 \leqslant m \leqslant n$ *and* $\gcd\{n, m\} = 1$*. Let* a *and* k *be two integers, with* $a > 1$*,* $k > 0$*. Prove that* k *divides* $\varphi(a^k - 1)$*.*

Problem 12 *Let* $f(z)$ *be analytic in the annulus* $\Omega = \{1 < |z| < 2\}$*. Assume that* f *has no zeros in* Ω*. Show that there exists an integer* n *and an analytic function* g *in* Ω *such that, for all* $z \in \Omega$*,* $f(z) = z^n e^{g(z)}$*.*

Problem 13 *Let* A *be an* $n \times n$ *real matrix,* A^t *its transpose. Show that* A^tA *and* A^t *have the same range. In other words, given* y*, show that the equation* $y = A^tAx$ *has a solution* x *if and only if the equation* $y = A^tz$ *has a solution* z*.*

Problem 14 *Let* $X \subset \mathbb{R}^n$ *be compact and let* $f : X \to \mathbb{R}$ *be continuous. Given* $\varepsilon > 0$*, show there is an* M *such that for all* $x, y \in X$*,*

$$|f(x) - f(y)| \leqslant M|x - y| + \varepsilon.$$

Problem 15 *Let* G_n *be the free group on* n *generators. Show that* G_2 *and* G_3 *are not isomorphic.*

Problem 16 *Prove that the polynomial*

$$p(z) = z^{47} - z^{23} + 2z^{11} - z^5 + 4z^2 + 1$$

has at least one root in the disc $|z| < 1$*.*

Problem 17 *Let V be a vector space of finite-dimension n over a field of characteristic 0. Prove that V is not the union of finitely many subspaces of dimension $n-1$.*

Problem 18 *Let the function f from $[0,1]$ to $[0,1]$ have the following properties:*

- *f is of class C^1;*
- *$f(0) = f(1) = 0$;*
- *f' is nonincreasing (i.e., f is concave).*

Prove that the arclength of the graph of f does not exceed 3.