Problem 1 Let $A$ be a finite abelian group, and $m$ the maximum of the orders of the elements of $A$. Put $S = \{a \in A \mid |a| = m\}$. Prove that $A$ is generated by $S$.

Problem 2 Let $f : [0, 1] \to \mathbb{R}$ be a real valued continuously differentiable function with $f(0) = 0$. Suppose also that there is a constant $M > 0$ such that, for $0 \leq x \leq 1$,

$$0 \leq f'(x) \leq M f(x).$$

Prove that $f(x) = 0$ for $0 \leq x \leq 1$.

Problem 3 Let $A$ be a real, upper-triangular, $n \times n$ matrix that commutes with its transpose. Prove that $A$ is diagonal.

Problem 4 Let $f(z)$ be analytic for $|z| < 1$ and suppose that

$$|f(z)| \leq \frac{1}{1 - |z|}.$$

Show that $|f'(0)| \leq 4$.

Problem 5 Let $G$ be a group, $G'$ its commutator subgroup, and $N$ a normal subgroup of $G$. Suppose that $N$ is cyclic. Prove that $g n = n g$ for all $g \in G'$ and all $n \in N$.

Problem 6 Let $X \subset \mathbb{R}^n$ be a closed set and $r$ a fixed positive real number. Let $Y = \{y \in \mathbb{R}^n \mid |x - y| = r \text{ for some } x \in X\}$. Show that $Y$ is closed.

Problem 7 Let $A$ and $B$ be diagonalizable linear transformations of $\mathbb{R}^n$ into itself such that $AB = BA$. Let $E$ be an eigenspace of $A$. Prove that the restriction of $B$ to $E$ is diagonalizable.

Problem 8 Evaluate the integral

$$I = \int_0^\infty \frac{\log x}{1 + x^2} dx.$$
Problem 9 Let $F$ be a field, $F[x]$ the polynomial ring in one variable over $F$, and $R$ a subring of $F[x]$ with $F \subset R$. Prove that there exists a finite set \{f_1, f_2, \ldots, f_n\} of elements of $F[x]$ such that $R = F[f_1, f_2, \ldots, f_n]$.

Problem 10 Let $\alpha$ be a number in $(0, 1)$. Prove that any sequence $(x_n)$ of real numbers satisfying the recurrence relation

$$x_{n+1} = \alpha x_n + (1 - \alpha) x_{n-1}$$

has a limit, and find an expression for the limit in terms of $\alpha$, $x_0$ and $x_1$.

Problem 11 Let $\varphi$ be Euler’s totient function; so if $n$ is a positive integer, then $\varphi(n)$ is the number of integers $m$ for which $1 \leq m \leq n$ and $\gcd\{n, m\} = 1$. Let $a$ and $k$ be two integers, with $a > 1$, $k > 0$. Prove that $k$ divides $\varphi(a^k - 1)$.

Problem 12 Let $f(z)$ be analytic in the annulus $\Omega = \{1 < |z| < 2\}$. Assume that $f$ has no zeros in $\Omega$. Show that there exists an integer $n$ and an analytic function $g$ in $\Omega$ such that, for all $z \in \Omega$, $f(z) = z^n e^{g(z)}$.

Problem 13 Let $A$ be an $n \times n$ real matrix, $A^t$ its transpose. Show that $A^t A$ and $A^t$ have the same range. In other words, given $y$, show that the equation $y = A^t A x$ has a solution $x$ if and only if the equation $y = A^t z$ has a solution $z$.

Problem 14 Let $X \subset \mathbb{R}^n$ be compact and let $f : X \to \mathbb{R}$ be continuous. Given $\varepsilon > 0$, show there is an $M$ such that for all $x, y \in X$,

$$|f(x) - f(y)| \leq M|x - y| + \varepsilon.$$

Problem 15 Let $G_n$ be the free group on $n$ generators. Show that $G_2$ and $G_3$ are not isomorphic.

Problem 16 Prove that the polynomial

$$p(z) = z^{47} - z^{23} + 2z^{11} - z^5 + 4z^2 + 1$$

has at least one root in the disc $|z| < 1$. 

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Problem 17 Let $V$ be a vector space of finite-dimension $n$ over a field of characteristic 0. Prove that $V$ is not the union of finitely many subspaces of dimension $n - 1$.

Problem 18 Let the function $f$ from $[0, 1]$ to $[0, 1]$ have the following properties:

- $f$ is of class $C^1$;
- $f(0) = f(1) = 0$;
- $f'$ is nonincreasing (i.e., $f$ is concave).

Prove that the arclength of the graph of $f$ does not exceed 3.