Problem 1 Let $R$ be a finite ring. Prove that there are positive integers $m$ and $n$ with $m > n$ such that $x^m = x^n$ for every $x$ in $R$.

Problem 2 Determine the group $\text{Aut}(\mathbb{C})$ of all one-to-one analytic maps of $\mathbb{C}$ onto $\mathbb{C}$.

Problem 3 Let the real valued functions $f_1, \ldots, f_{n+1}$ on $\mathbb{R}$ satisfy the system of differential equations
\[
\begin{align*}
f'_k + f'_k &= (k+1)f_{k+1} - kf_k, \quad k = 1, \ldots, n \\
f'_n &= -(n+1)f_{n+1}.
\end{align*}
\]
Prove that for each $k$,
\[
\lim_{t \to \infty} f_k(t) = 0.
\]

Problem 4 Find the Jordan Canonical Form of the matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

Problem 5 Let $f$ be a continuous, strictly increasing function from $[0, \infty)$ onto $[0, \infty)$ and let $g = f^{-1}$. Prove that
\[
\int_0^a f(x) \, dx + \int_0^b g(y) \, dy \geq ab
\]
for all positive numbers $a$ and $b$, and determine the condition for equality.

Problem 6 Let $f$ be a function from $[0, 1]$ into itself whose graph
\[
G_f = \{(x, f(x)) \mid x \in [0, 1]\}
\]
is a closed subset of the unit square. Prove that $f$ is continuous.

Note: See also Problem ??.
Problem 7 Find all abelian groups of order 8, up to isomorphism. Then identify which type occurs in each of

1. \((\mathbb{Z}_{15})^*\),
2. \((\mathbb{Z}_{17})^*/(\pm 1)\),
3. the roots of \(z^8 - 1\) in \(\mathbb{C}\),
4. \(\mathbb{F}_8^+\),
5. \((\mathbb{Z}_{16})^*\).

\(\mathbb{F}_8\) is the field of eight elements, and \(\mathbb{F}_8^+\) is its underlying additive group; \(R^*\) is the group of invertible elements in the ring \(R\), under multiplication.

Problem 8 Do the functions \(f(z) = e^z + z\) and \(g(z) = ze^z + 1\) have the same number of zeros in the strip \(-\frac{\pi}{2} < \Im z < \frac{\pi}{2}\)?

Problem 9 Let \(A\) and \(B\) be real symmetric \(n \times n\) matrices. Assume that the eigenvalues of \(A\) all lie in the interval \([a_1, a_2]\) and those of \(B\) all lie in the interval \([b_1, b_2]\). Prove that the eigenvalues of \(A + B\) all lie in the interval \([a_1 + b_1, a_2 + b_2]\).

Problem 10 Find (up to isomorphism) all groups of order \(2p\), where \(p\) is a prime \((p \geq 2)\).

Problem 11 Let \(f\) be an analytic function on a disc \(D\) whose center is the point \(z_0\). Assume that \(|f'(z) - f'(z_0)| < |f'(z_0)|\) on \(D\). Prove that \(f\) is one-to-one on \(D\).

Problem 12 Let \(n\) be a positive integer and let \(f\) be a polynomial in \(\mathbb{R}[x]\) of degree \(n\). Prove that there are real numbers \(a_0, a_1, \ldots, a_n\), not all equal to zero, such that the polynomial

\[
\sum_{i=0}^{n} a_i x^{2^i}
\]

is divisible by \(f\).
Problem 13 Let $A$ be a complex $n \times n$ matrix, and let $C(A)$ be the commutant of $A$; that is, the set of complex $n \times n$ matrices $B$ such that $AB = BA$. (It is obviously a subspace of $M_{n\times n}$, the vector space of all complex $n \times n$ matrices.) Prove that $\dim C(A) \geq n$.

Problem 14 Let the group $G$ be generated by two elements, $a$ and $b$, both of order 2. Prove that $G$ has a subgroup of index 2.

Problem 15 Prove that a real valued $C^3$ function $f$ on $\mathbb{R}^2$ whose Laplacian,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

is everywhere positive cannot have a local maximum.

Problem 16 Let $n$ be a positive integer. Prove that the polynomial

$$f(x) = \sum_{i=0}^{n} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$$

in $\mathbb{R}[x]$ has $n$ distinct complex zeros, $z_1, z_2, \ldots, z_n$, and that they satisfy

$$\sum_{i=1}^{n} z_i^{-j} = 0 \quad \text{for} \quad 2 \leq j \leq n.$$

Problem 17 Prove that

$$\int_0^\infty \frac{x}{e^x - e^{-x}} \, dx = \frac{\pi^2}{8}.$$

Problem 18 Let $g$ be a continuous real valued function on $[0, 1]$. Prove that there exists a continuous real valued function $f$ on $[0, 1]$ satisfying the equation

$$f(x) - \int_0^x f(x-t)e^{-t^2} \, dt = g(x).$$