Preliminary Exam - Fall 1988

Problem 1 *Let R be a finite ring. Prove that there are positive integers m and n with $m > n$ such that $x^m = x^n$ for every x in R.*

Problem 2 *Determine the group* $\mathrm{Aut}(\mathbb{C}\,)$ *of all one-to-oneanalytic maps of* $\mathbb{C}$ *onto* $\mathbb{C}$ *.*

Problem 3 *Let the real valued functions $f_1, \dots, f_{n+1}$ on $\mathbb{R}$ satisfy the system of differential equations*

$$\begin{aligned} f'_{k+1} + f'_k &= (k+1)f_{k+1} - kf_k, \quad k = 1, \dots, n \\ f'_{n+1} &= -(n+1)f_{n+1}. \end{aligned}$$

Prove that for each k,

$$\lim_{t \to \infty} f_k(t) = 0.$$

Problem 4 *Find the Jordan Canonical Form of the matrix*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Problem 5 *Let f be a continuous, strictly increasing function from $[0, \infty)$ onto $[0, \infty)$ and let $g = f^{-1}$. Prove that*

$$\int_0^a f(x)\, dx + \int_0^b g(y)\, dy \geqslant ab$$

for all positive numbers a and b, and determine the condition for equality.

Problem 6 *Let f be a function from $[0, 1]$ into itself whose graph*

$$G_f = \{(x, f(x)) \mid x \in [0, 1]\}$$

is a closed subset of the unit square. Prove that f is continuous.
Note: See also Problem **??**.

Problem 7 *Find all abelian groups of order 8, up to isomorphism. Then identify which type occurs in each of*

1. $(\mathbb{Z}_{15})^*$,
2. $(\mathbb{Z}_{17})^* / (\pm 1)$,
3. *the roots of* $z^8 - 1$ *in* $\mathbb{C}$,
4. $\mathbf{F}_8^+$,
5. $(\mathbb{Z}_{16})^*$.

$\mathbf{F}_8$ *is the field of eight elements, and* $\mathbf{F}_8^+$ *is its underlying additive group;* R^* *is the group of invertible elements in the ring* R*, under multiplication.*

Problem 8 *Do the functions* $f(z) = e^z + z$ *and* $g(z) = ze^z + 1$ *have the same number of zeros in the strip* $-\frac{\pi}{2} < \Im z < \frac{\pi}{2}$ *?*

Problem 9 *Let* A *and* B *be real symmetric* $n \times n$ *matrices. Assume that the eigenvalues of* A *all lie in the interval* $[a_1, a_2]$ *and those of* B *all lie in the interval* $[b_1, b_2]$*. Prove that the eigenvalues of* $A + B$ *all lie in the interval* $[a_1 + b_1, a_2 + b_2]$*.*

Problem 10 *Find (up to isomorphism) all groups of order* $2p$*, where* p *is a prime* ($p \geqslant 2$)*.*

Problem 11 *Let* f *be an analytic function on a disc* D *whose center is the point* z_0*. Assume that* $|f'(z) - f'(z_0)| < |f'(z_0)|$ *on* D*. Prove that* f *is one-to-one on* D*.*

Problem 12 *Let* n *be a positive integer and let* f *be a polynomial in* $\mathbb{R}[x]$ *of degree* n*. Prove that there are real numbers* $a_0, a_1, \ldots, a_n$*, not all equal to zero, such that the polynomial*

$$\sum_{i=0}^{n} a_i x^{2^i}$$

is divisible by f*.*

Problem 13 *Let A be a complex $n \times n$ matrix, and let $C(A)$ be the* commutant *of A; that is, the set of complex $n \times n$ matrices B such that$AB = BA$. (It is obviously a subspace of $M_{n\times n}$, the vector space of all complex $n \times n$ matrices.) Prove that $\dim C(A) \geqslant n$.*

Problem 14 *Let the group G be generated by two elements, a and b, both of order 2. Prove that G has a subgroup of index 2.*

Problem 15 *Prove that a real valued C^3 function f on $\mathbb{R}^2$ whose Laplacian,*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

is everywhere positive cannot have a local maximum.

Problem 16 *Let n be a positive integer. Prove that the polynomial*

$$f(x) = \sum_{i=0}^{n} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$$

in $\mathbb{R}[x]$ has n distinct complex zeros, $z_1, z_2, \ldots, z_n$, and that they satisfy

$$\sum_{i=1}^{n} z_i^{-j} = 0 \quad for \quad 2 \leqslant j \leqslant n.$$

Problem 17 *Prove that*

$$\int_0^\infty \frac{x}{e^x - e^{-x}}\,dx = \frac{\pi^2}{8}\cdot$$

Problem 18 *Let g be a continuous real valued function on $[0, 1]$. Prove that there exists a continuous real valued function f on $[0, 1]$ satisfying the equation*

$$f(x) - \int_0^x f(x-t)e^{-t^2}\,dt = g(x).$$