Preliminary Exam - Fall 1987

Problem 1 *Prove that* $(\cos\theta)^p \leqslant \cos(p\,\theta)$ *for* $0 \leqslant \theta \leqslant \pi/2$ *and* $0 < p < 1$*.*

Problem 2 *Suppose that* $\{f_n\}$ *is a sequence of nondecreasing functions which map the unit interval into itself. Suppose that*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

pointwise and that f *is a continuous function. Prove that* $f_n(x) \to f(x)$ *uniformly as* $n \to \infty$, $0 \leqslant x \leqslant 1$*. Note that the functions* f_n *are not necessarily continuous.*

Problem 3 *Show that the following limit exists and is finite:*

$$\lim_{t\to 0^+}\left(\int_0^1 \frac{dx}{(x^4+t^4)^{1/4}} + \log t\right).$$

Problem 4 *Let* u *and* v *be two real valued* C^1 *functions on* $\mathbb{R}^2$ *such that the gradient* ∇u *is never* 0*, and such that, at each point,* ∇v *and* ∇u *are linearly dependent vectors. Given* $p_0 = (x_0, y_0) \in \mathbb{R}^2$*, show that there is a* C^1 *function* F *of one variable such that* $v(x,y) = F\left(u(x,y)\right)$ *in some neighborhood of* p_0*.*

Problem 5 *Calculate* A^{100} *and* A^{-7}*, where*

$$A = \begin{pmatrix} 3/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

Problem 6 *Let* G *and* H *be finite groups of relatively prime order. Show that* $\mathrm{Aut}(G\times H)$*, the group of automorphisms of* $G \times H$*, is isomorphic to the direct product of* $\mathrm{Aut}(G)$ *and* $\mathrm{Aut}(H)$*.*

Problem 7 *Let* A *and* B *be real* $n\times n$ *symmetric matrices with* B *positive definite. Consider the function defined for* $x \neq 0$ *by*

$$G(x) = \frac{\langle Ax, x\rangle}{\langle Bx, x\rangle}\cdot$$

1. *Show that G attains its maximum value.*

2. *Show that any maximum point U for G is an eigenvector for a certain matrix related to A and B and show which matrix.*

Problem 8 *Let R be the set of 2×2 matrices of the form*

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

where a, b are elements of a given field $\mathbf{F}$. Show that with the usual matrix operations, R is a commutative ring with identity. For which of the following fields $\mathbf{F}$ is R a field: $F = \mathbb{Q}\,, \mathbb{C}\,, \mathbb{Z}_5, \mathbb{Z}_7$?

Problem 9 *Evaluate the integral*

$$I = \int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\, d\theta\,.$$

Problem 10 *If $f(z)$ is analytic in the open disc $|z| < 1$, and$|f(z)| \leqslant 1/(1-|z|)$, show that*

$$|a_n| = \left|\frac{f^{(n)}(0)}{n!}\right| \leqslant (n+1)\left(1+\frac{1}{n}\right)^n < e(n+1).$$

Problem 11 *Let V be a finite-dimensional vector space and $T{:}V \to V$ a diagonalizable linear transformation. Let $W \subset V$ be a linear subspace which is mapped into itself by T. Show that the restriction of T to W is diagonalizable.*

Problem 12 *Given two* real *$n \times n$ matrices A and B, suppose that there is a nonsingular* complex *matrix C such that $CAC^{-1} = B$. Show that there exists a* real *nonsingular $n \times n$ matrix C with this property.*

Problem 13 *Let A be the group of rational numbers under addition, and let M be the group of positive rational numbers under multiplication. Determine all homomorphisms $\varphi : A \to M$.*

Problem 14 *Show that $M_{n\times n}(\mathbf{F})$, the ring of all $n \times n$ matrices over the field $\mathbf{F}$, has no proper two sided ideals.*

Problem 15 *Let $f(z)$ be analytic for $z \neq 0$, and suppose that$f(1/z) = f(z)$. Suppose also that $f(z)$ is real for all z on the unit circle $|z| = 1$. Prove that $f(z)$ is real for all real $z \neq 0$.*

Problem 16 *How many zeros (counting multiplicities) does the polynomial*

$$2z^5 - 6z^3 + z + 1$$

have in the annular region $1 \leqslant |z| \leqslant 2$?

Problem 17 *Let u be a positive* harmonic *function on $\mathbb{R}^2$; that is,*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Show that u is constant.

Problem 18 *Find a curve C in $\mathbb{R}^2$, passing through the point $(3, 2)$, with the following property: Let $L(x_0, y_0)$ be the segment of the tangent line to C at (x_0, y_0) which lies in the first quadrant. Then each point (x_0, y_0) of C is the midpoint of $L(x_0, y_0)$.*

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Problem 19 *Define a sequence of positive numbers as follows. Let$x_0 > 0$ be any positive number, and let $x_{n+1} = (1 + x_n)^{-1}$. Prove that this sequence converges, and find its limit.*

Problem 20 *Let S be the set of all real C^1 functions f on $[0, 1]$ such that $f(0) = 0$ and*

$$\int_0^1 f'(x)^2 \, dx \leqslant 1.$$

Define

$$J(f) = \int_0^1 f(x) \, dx \, .$$

Show that the function J is bounded on S, and compute its supremum. Is there a function $f_0 \in S$ at which J attains its maximum value? If so, what is f_0?