Problem 1  Prove that \( (\cos \theta)^p \leq \cos(p \theta) \) for \( 0 \leq \theta \leq \pi/2 \) and \( 0 < p < 1 \).

Problem 2  Suppose that \( \{f_n\} \) is a sequence of nondecreasing functions which map the unit interval into itself. Suppose that
\[
\lim_{n \to \infty} f_n(x) = f(x)
\]
pointwise and that \( f \) is a continuous function. Prove that \( f_n(x) \to f(x) \) uniformly as \( n \to \infty \), \( 0 \leq x \leq 1 \). Note that the functions \( f_n \) are not necessarily continuous.

Problem 3  Show that the following limit exists and is finite:
\[
\lim_{t \to 0^+} \left( \int_0^1 \frac{dx}{(x^4 + t^4)^{1/4}} + \log t \right).
\]

Problem 4  Let \( u \) and \( v \) be two real valued \( C^1 \) functions on \( \mathbb{R}^2 \) such that the gradient \( \nabla u \) is never 0, and such that, at each point, \( \nabla v \) and \( \nabla u \) are linearly dependent vectors. Given \( p_0 = (x_0, y_0) \in \mathbb{R}^2 \), show that there is a \( C^1 \) function \( F \) of one variable such that \( v(x, y) = F(u(x, y)) \) in some neighborhood of \( p_0 \).

Problem 5  Calculate \( A^{100} \) and \( A^{-7} \), where
\[
A = \begin{pmatrix} 3/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}.
\]

Problem 6  Let \( G \) and \( H \) be finite groups of relatively prime order. Show that \( \text{Aut}(G \times H) \), the group of automorphisms of \( G \times H \), is isomorphic to the direct product of \( \text{Aut}(G) \) and \( \text{Aut}(H) \).

Problem 7  Let \( A \) and \( B \) be real \( n \times n \) symmetric matrices with \( B \) positive definite. Consider the function defined for \( x \neq 0 \) by
\[
G(x) = \frac{\langle Ax, x \rangle}{\langle Bx, x \rangle}.
\]
1. Show that $G$ attains its maximum value.

2. Show that any maximum point $U$ for $G$ is an eigenvector for a certain matrix related to $A$ and $B$ and show which matrix.

**Problem 8** Let $R$ be the set of $2 \times 2$ matrices of the form
\[
\begin{pmatrix}
    a & -b \\
    b & a
\end{pmatrix}
\]
where $a, b$ are elements of a given field $F$. Show that with the usual matrix operations, $R$ is a commutative ring with identity. For which of the following fields $F$ is $R$ a field: $F = \mathbb{Q}, \mathbb{C}, \mathbb{Z}_5, \mathbb{Z}_7$?

**Problem 9** Evaluate the integral
\[
I = \int_{0}^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta} \, d\theta.
\]

**Problem 10** If $f(z)$ is analytic in the open disc $|z| < 1$, and $|f(z)| \leq 1/(1 - |z|)$, show that
\[
|a_n| = \left| \frac{f^{(n)}(0)}{n!} \right| \leq (n + 1) \left( 1 + \frac{1}{n} \right)^n < e(n + 1).
\]

**Problem 11** Let $V$ be a finite-dimensional vector space and $T: V \to V$ a diagonalizable linear transformation. Let $W \subset V$ be a linear subspace which is mapped into itself by $T$. Show that the restriction of $T$ to $W$ is diagonalizable.

**Problem 12** Given two real $n \times n$ matrices $A$ and $B$, suppose that there is a nonsingular complex matrix $C$ such that $CAC^{-1} = B$. Show that there exists a real nonsingular $n \times n$ matrix $C$ with this property.

**Problem 13** Let $A$ be the group of rational numbers under addition, and let $M$ be the group of positive rational numbers under multiplication. Determine all homomorphisms $\varphi : A \to M$.

**Problem 14** Show that $M_{n \times n}(F)$, the ring of all $n \times n$ matrices over the field $F$, has no proper two sided ideals.
Problem 15 Let $f(z)$ be analytic for $z \neq 0$, and suppose that $f(1/z) = f(z)$. Suppose also that $f(z)$ is real for all $z$ on the unit circle $|z| = 1$. Prove that $f(z)$ is real for all real $z \neq 0$.

Problem 16 How many zeros (counting multiplicities) does the polynomial $2z^5 - 6z^3 + z + 1$ have in the annular region $1 \leq |z| \leq 2$?

Problem 17 Let $u$ be a positive harmonic function on $\mathbb{R}^2$; that is,
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.
\]
Show that $u$ is constant.

Problem 18 Find a curve $C$ in $\mathbb{R}^2$, passing through the point $(3,2)$, with the following property: Let $L(x_0, y_0)$ be the segment of the tangent line to $C$ at $(x_0, y_0)$ which lies in the first quadrant. Then each point $(x_0, y_0)$ of $C$ is the midpoint of $L(x_0, y_0)$.

Problem 19 Define a sequence of positive numbers as follows. Let $x_0 > 0$ be any positive number, and let $x_{n+1} = (1 + x_n)^{-1}$. Prove that this sequence converges, and find its limit.

Problem 20 Let $S$ be the set of all real $C^1$ functions $f$ on $[0, 1]$ such that $f(0) = 0$ and
\[
\int_0^1 f'(x)^2 \, dx \leq 1.
\]
Define
\[
J(f) = \int_0^1 f(x) \, dx.
\]
Show that the function $J$ is bounded on $S$, and compute its supremum. Is there a function $f_0 \in S$ at which $J$ attains its maximum value? If so, what is $f_0$?