Preliminary Exam - Fall 1986

Problem 1 *The Arzelà–Ascoli Theorem asserts that the sequence* $\{f_n\}$ *of continuous real valued functions on a metric space* Ω *is precompact (i.e., has a uniformly convergent subsequence) if*

(i) Ω *is compact,*

(ii) $\sup \|f_n\| < \infty$ *(where* $\|f_n\| = \sup\{|f_n(x)| \mid x \in \Omega\}$*),*

(iii) the sequence is equicontinuous.

Give examples of sequences which are not precompact such that: (i) and (ii) hold but (iii) fails; (i) and (iii) hold but (ii) fails; (ii) and (iii) hold but (i) fails. Take Ω *to be a subset of the real line. Sketch the graph of a typical member of the sequence in each case.*

Problem 2 *Let the points* a*,* b*, and* c *lie on the unit circle of the complex plane and satisfy* $a + b + c = 0$*. Prove that* a*,* b*, and* c *form the vertices of an equilateral triangle.*

Problem 3 *Evaluate*

$$\iint_{\mathcal{R}} (x^3 - 3xy^2)\, dxdy\,,$$

where

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid (x+1)^2 + y^2 \leqslant 9, \quad (x-1)^2 + y^2 \geqslant 1\}.$$

Problem 4 *Show that the polynomial* $p(z) = z^5 - 6z + 3$ *has five distinct complex roots, of which exactly three (and not five) are real.*

Problem 5 *Let* $M_{2\times 2}$ *denote the vector space of complex 2×2 matrices. Let*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and let the linear transformation $T : M_{2\times 2} \to M_{2\times 2}$ *be defined by*$T(X) = XA - AX$*. Find the Jordan Canonical Form for* T*.*

Problem 6 *Prove the following theorem, or find a counterexample: If* p *and* q *are continuous real valued functions on* $\mathbb{R}$ *such that* $|q(x)| \leqslant |p(x)|$ *for all* x*, and if every solution* f *of the differential equation*

$$f' + qf = 0$$

satisfies $\lim_{x\to+\infty} f(x) = 0$*, then every solution* f *of the differential equation*

$$f' + pf = 0$$

satisfies $\lim_{x\to+\infty} f(x) = 0$.

Problem 7 *Let* $\mathbf{F}$ *be a field containing* $\mathbb{Q}$ *such that* $[\mathbf{F} : \mathbb{Q}\,] = 2$*. Prove that there exists a unique integer* m *such that* m *has no multiple prime factors and* $\mathbf{F}$ *is isomorphic to* $\mathbb{Q}\,(\sqrt{m}\,)$.

Problem 8 *Let* f *be a continuous real valued function on* $[0,1]$ *such that, for each* $x_0 \in [0,1)$,

$$\limsup_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} \geqslant 0.$$

Prove that f *is nondecreasing.*

Problem 9 *Evaluate*

$$\int_0^\infty \frac{\log x}{(x^2+1)(x^2+4)}\, dx\,.$$

Problem 10 *For* f *a real valued function on the real line, define the function* $\triangle f$ *by* $\triangle f(x) = f(x+1) - f(x)$*. For* $n \geqslant 2$*, define* $\triangle^n f$ *recursively by* $\triangle^n f = \triangle(\triangle^{n-1} f)$*. Prove that* $\triangle^n f = 0$ *if and only if* f *has the form*$f(x) = a_0(x) + a_1(x)x + \cdots + a_{n-1}(x)x^{n-1}$ *where* $a_0, a_1, \ldots, a_{n-1}$ *are periodic functions of period* 1.

Problem 11 *Let* A *be an* $m \times n$ *matrix with entries in a field* $\mathbf{F}$*. Define the row rank and the column rank of* A *and show from first principles that they are equal.*

Problem 12 *Let* $\{U_1, U_2, \ldots\}$ *be a cover of* $\mathbb{R}^n$ *by open sets. Prove that there is a cover* $\{V_1, V_2, \ldots\}$ *such that*

1. $V_j \subset U_j$ *for each* j;

2. *each compact subset of* $\mathbb{R}^n$ *is disjoint from all but finitely many of the* V_j.

Problem 13 *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be defined by:*

$$f(x, y) = \begin{cases} x^{4/3} \sin(y/x) & if \quad x \neq 0 \\ 0 & if \quad x = 0\,. \end{cases}$$

Determine all points at which f *is differentiable.*

Problem 14 *Let* a *and* b *be real numbers. Prove that there are two orthogonal unit vectors* u *and* v *in* $\mathbb{R}^3$ *such that* $u = (u_1, u_2, a)$ *and* $v = (v_1, v_2, b)$ *if and only if* $a^2 + b^2 \leqslant 1$.

Problem 15 *Prove that if* p *is a prime number* (> 0) *then the polynomial*

$$f(x) = x^{p-1} + x^{p-2} + \cdots + 1$$

is irreducible in $\mathbb{Q}[x]$.

Problem 16 *Discuss the solvability of the differential equation*

$$(e^x \sin y)(y')^3 + (e^x \cos y)y' + e^y \tan x = 0$$

with the initial condition $y(0) = 0$. *Does a solution exist in some interval about* 0? *If so, is it unique?*

Problem 17 *Let* G *be a subgroup of* S_5, *the group of all permutations of five objects Prove that if* G *contains a* 5*-cycle and a* 2*-cycle, then* $G = S_5$.

Problem 18 *Evaluate*

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{z^{11}}{12z^{12} - 4z^9 + 2z^6 - 4z^3 + 1}\, dz$$

where the direction of integration is counterclockwise.

Problem 19 *Prove that if six people are riding together in an Evans Hall elevator, there is either a three-person subset of mutual friends (each knows the other two) or a three-person subset of mutual strangers (each knows neither of the other two).*

Problem 20 *Let f be a real valued continuous function on $[0, \infty)$ such that*

$$\lim_{x \to \infty} \left(f(x) + \int_0^x f(t)\, dt \right)$$

exists. Prove that

$$\lim_{x \to \infty} f(x) = 0.$$