Preliminary Exam - Fall 1985

Problem 1 *Evaluate the integral*

$$\int_0^\infty \frac{1-\cos ax}{x^2}\,dx$$

for $a \in \mathbb{R}$.

Problem 2 *Prove that for every* $\lambda > 1$, *the equation* $ze^{\lambda - z} = 1$ *has exactly one root in the disc* $|z| < 1$ *and that this root is real.*

Problem 3

1. *How many different monic irreducible polynomials of degree* 2 *are there over the field* $\mathbb{Z}_5$*?*
2. *How many different monic irreducible polynomials of degree* 3 *are there over the field* $\mathbb{Z}_5$*?*

Problem 4 *Let* G *be a finite subgroup of the group* $\mathbb{C}^*$ *of nonzero complex numbers under multiplication. Prove that* G *is cyclic.*

Problem 5 *How many roots does the polynomial* $z^4 + 3z^2 + z + 1$ *have in the right half* z*-plane?*

Problem 6 *Let* k *be real,* n *an integer* $\geqslant 2$, *and let* $A = (a_{ij})$ *be the* $n \times n$ *matrix such that all diagonal entries* $a_{ii} = k$, *all entries* $a_{i\,i\pm1}$ *immediately above or below the diagonal equal* 1, *and all other entries equal* 0. *For example, if* $n = 5$,

$$A = \begin{pmatrix} k & 1 & 0 & 0 & 0 \\ 1 & k & 1 & 0 & 0 \\ 0 & 1 & k & 1 & 0 \\ 0 & 0 & 1 & k & 1 \\ 0 & 0 & 0 & 1 & k \end{pmatrix}.$$

Let λ_{min} *and* λ_{max} *denote the smallest and largest eigenvalues of* A, *respectively. Show that* $\lambda_{min} \leqslant k - 1$ *and* $\lambda_{max} \geqslant k + 1$.

Problem 7 *Let $y(t)$ be a real valued solution, defined for $0 < t < \infty$, of the differential equation*

$$\frac{dy}{dt} = e^{-y} - e^{-3y} + e^{-5y}.$$

Show that $y(t) \to +\infty$ as $t \to +\infty$.

Problem 8 *Let $f(x)$, $0 \leqslant x \leqslant 1$, be a real valued continuous function. Show that*

$$\lim_{n\to\infty} (n+1) \int_0^1 x^n f(x)\, dx = f(1).$$

Problem 9 *Let A be the symmetric matrix*

$$\frac{1}{6}\begin{pmatrix} 13 & -5 & -2 \\ -5 & 13 & -2 \\ -2 & -2 & 10 \end{pmatrix}.$$

Denote by v the column vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

in $\mathbb{R}^3$, and by x^t its transpose (x, y, z). Let $\|v\|$ denote the length of the vector v. As v ranges over the set of vectors for which $v^t Av = 1$, show that $\|v\|$ is bounded, and determine its least upper bound.

Problem 10 *Let f and f_n, $n = 1, 2, \ldots$, be functions from $\mathbb{R}$ to $\mathbb{R}$. Assume that $f_n(x_n) \to f(x)$ as $n \to \infty$ whenever $x_n \to x$. Show that f is continuous. Note: The functions f_n are not assumed to be continuous.*

Problem 11 *Let G be a subgroup of the symmetric group on six objects, S_6. Assume that G has an element of order 6. Prove that G has a normal subgroup H of index 2.*

Problem 12 *Evaluate*

$$\int_0^{2\pi} e^{e^{i\theta}}\, d\theta\,.$$

Problem 13 *Let $f(z)$ be analytic on the right half-plane$H = \{z \mid \Re z > 0\}$ and suppose $|f(z)| \leqslant 1$ for $z \in H$. Suppose also that $f(1) = 0$. What is the largest possible value of $|f'(1)|$?*

Problem 14 *Suppose that A and B are endomorphisms of a finite-dimensional vector space V over a field $\mathbf{F}$. Prove or disprove the following statements:*

1. *Every eigenvector of AB is also an eigenvector of BA.*
2. *Every eigenvalue of AB is also an eigenvalue of BA.*

Problem 15 *Let $0 \leqslant a \leqslant 1$ be given. Determine all nonnegative continuous functions f on $[0, 1]$ which satisfy the following three conditions:*

$$\int_0^1 f(x)\,dx = 1,$$

$$\int_0^1 x f(x)\,dx = a,$$

$$\int_0^1 x^2 f(x)\,dx = a^2.$$

Problem 16 *Let $f(x) = x^5 - 8x^3 + 9x - 3$ and $g(x) = x^4 - 5x^2 - 6x + 3$. Prove that there is an integer d such that the polynomials $f(x)$ and $g(x)$ have a common root in the field $\mathbb{Q}(\sqrt{d})$. What is d?*

Problem 17 *Let (M, d) be a nonempty complete metric space. Let S map M into M, and write S^2 for $S \circ S$; that is, $S^2(x) = S(S(x))$. Suppose that S^2 is a* strict contraction*; that is, there is a constant $\lambda < 1$ such that for all points $x, y \in M$, $d(S^2(x), S^2(y)) \leqslant \lambda d(x, y)$. Show that S has a unique fixed point in M.*

Problem 18 *Let G be a group. For any subset X of G, define its* centralizer *$C(X)$ to be $\{y \in G \mid xy = yx, \text{ for all } x \in X\}$. Prove the following:*

1. *If $X \subset Y$, then $C(Y) \subset C(X)$.*
2. *$X \subset C(C(X))$.*
3. *$C(X) = C(C(C(X)))$.*

Problem 19 *An* $n\times n$ *real matrix* T *is* positive definite *if* T *is symmetric and* $\langle Tx, x\rangle > 0$ *for all nonzero vectors* $x \in \mathbb{R}^n$*, where* $\langle u, v\rangle$ *is the standard inner product. Suppose that* A *and* B *are two positive definite real matrices.*

1. *Show that there is a basis* $\{v_1, v_2, \ldots, v_n\}$ *of* $\mathbb{R}^n$ *and real numbers* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *such that, for* $1 \leqslant i, j \leqslant n$*:*

$$\langle Av_i, v_j\rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

   and

$$\langle Bv_i, v_j\rangle = \begin{cases} \lambda_i & i = j \\ 0 & i \neq j \end{cases}$$

2. *Deduce from Part 1 that there is an invertible real matrix* U *such that* $U^t AU$ *is the identity matrix and* $U^t BU$ *is diagonal.*

Problem 20 *Let* f *be a differentiable function on* $[0, 1]$ *and let*

$$\sup_{0<x<1} |f'(x)| = M < \infty.$$

Let n *be a positive integer. Prove that*

$$\left|\sum_{j=0}^{n-1} \frac{f(j/n)}{n} - \int_0^1 f(x)\, dx\right| \leqslant \frac{M}{2n}.$$