Preliminary Exam - Fall 1984

Problem 1 *Let G be a group and H a subgroup of index $n < \infty$. Prove or disprove the following statements:*

1. *If $a \in G$, then $a^n \in H$.*
2. *If $a \in G$, then for some k, $1 \leqslant k \leqslant n$, we have $a^k \in H$.*

Problem 2 *Let A and B be $n \times n$ real matrices, and k a positive integer. Find*

1.
$$\lim_{t \to 0} \frac{1}{t} \left((A + tB)^k - A^k \right).$$

2.
$$\left. \frac{d}{dt} \operatorname{tr} (A + tB)^k \right|_{t=0}.$$

Problem 3 *Prove or supply a counterexample: If f is a nondecreasing real valued function on $[0,1]$, then there is a sequence of continuous functions on $[0,1]$, $\{f_n\}$, such that for each $x \in [0,1]$,*

$$\lim_{n \to \infty} f_n(x) = f(x).$$

Problem 4 *Evaluate*

$$\int_0^\infty \frac{x - \sin x}{x^3} \, dx \, .$$

Problem 5 *Consider the differential equation*

$$\frac{dy}{dx} = 3xy + \frac{y}{1 + y^2}.$$

Prove

1. *For each* $n = 1, 2, \ldots$, *there is a unique solution* $y = f_n(x)$ *defined for* $0 \leqslant x \leqslant 1$ *such that* $f_n(0) = 1/n$.

2. $\lim\limits_{n\to\infty} f_n(1) = 0$.

Problem 6 *Let* gcd *abbreviate greatest common divisor and* lcm *abbreviate least common multiple. For three nonzero integers* a, b, c, *show that*

$$\gcd\{a, \operatorname{lcm}\{b, c\}\} = \operatorname{lcm}\{\gcd\{a, b\}, \gcd\{a, c\}\}.$$

Problem 7 *Let* $\mathbb{R}[x_1, \ldots, x_n]$ *be the polynomial ring over the real field* $\mathbb{R}$ *in the* n *variables* $x_1, \ldots, x_n$. *Let the matrix* A *be the* $n \times n$ *matrix whose* i^{th} *row is* $(1, x_i, x_i^2, \ldots, x_i^{n-1})$, $i = 1, \ldots, n$. *Show that*

$$\det A = \prod_{i>j}(x_i - x_j).$$

Problem 8 *Let* a, b, c, *and* d *be real numbers, not all zero. Find the eigenvalues of the following* 4×4 *matrix and describe the eigenspace decomposition of* $\mathbb{R}^4$:

$$\begin{pmatrix} aa & ab & ac & ad \\ ba & bb & bc & bd \\ ca & cb & cc & cd \\ da & db & dc & dd \end{pmatrix}.$$

Problem 9 *Let* f *and* g *be analytic functions in the open unit disc, and let* C_r *denote the circle with center* 0 *and radius* r, *oriented counterclockwise.*

1. *Prove that the integral*

$$\frac{1}{2\pi i}\int_{C_r} \frac{1}{w} f(w) g\left(\frac{z}{w}\right)\, dw$$

   is independent of r *as long as* $|z| < r < 1$ *and that it defines an analytic function* $h(z)$, $|z| < 1$.

2. *Prove or supply a counterexample: If* $f \not\equiv 0$ *and* $g \not\equiv 0$, *then* $h \not\equiv 0$.

Problem 10 *Prove or supply a counterexample:*

1. *If f and g are C^1 real valued functions on $(0,1)$, if*

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0,$$

if g and g' never vanish, and if

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = c,$$

then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = c.$$

2. *Do the same question for complex valued f and g.*

Problem 11 *Show that all groups of order $\leqslant 5$ are commutative. Give an example of a noncommutative group of order 6.*

Problem 12 *Let θ and φ be fixed, $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant \varphi \leqslant 2\pi$ and let R be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ whose matrix in the standard basis $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ is*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

Let S be the linear transformation of $\mathbb{R}^3$ to $\mathbb{R}^3$ whose matrix with respect to the basis

$$\left\{ \frac{1}{\sqrt{2}}(\vec{\imath} + \vec{k}), \vec{\jmath}, \frac{1}{\sqrt{2}}(\vec{\imath} - \vec{k}) \right\}$$

is

$$\begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove that $T = R \circ S$ leaves a line invariant.

Problem 13 *Show that if f is a homeomorphism of $[0,1]$ onto itself, then there is a sequence $\{p_n\}$, $n = 1, 2, 3, \ldots$ of polynomials such that $p_n \to f$ uniformly on $[0,1]$ and each p_n is a homeomorphism of $[0,1]$ onto itself.*

Problem 14 *Evaluate*

1.

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x+x^2)^2}$$

2.

$$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} \quad \textit{where} \quad a > 1.$$

Problem 15 *Consider the differential equation*

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -ay - x^3 - x^5, \quad \textit{where} \;\; a > 0.$$

1. *Show that*

$$F(x,y) = \frac{y^2}{2} + \frac{x^4}{4} + \frac{x^6}{6}$$

decreases along solutions.

2. *Show that for any* $\varepsilon > 0$*, there is a* $\delta > 0$ *such that whenever* $\| (x(0), y(0)) \| < \delta$*, there is a unique solution* $(x(t), y(t))$ *of the given equations with the initial condition* $(x(0), y(0))$ *which is defined for all* $t \geqslant 0$ *and satisfies* $\| (x(t), y(t)) \| < \varepsilon$.

Problem 16 *Let* a *be an element in a field* $\mathbf{F}$ *and let* p *be a prime. Assume* a *is not a* p^{th} *power. Show that the polynomial* $x^p - a$ *is irreducible in* $\mathbf{F}[x]$.

Problem 17 *Let* M *be the* $n \times n$ *matrix over a field* $\mathbf{F}$ *all of whose entries are equal to* 1*. Find the Jordan Canonical Form of* M *and discuss the extent to which the Jordan form depends on the characteristic of the field* $\mathbf{F}$.

Problem 18 *Let* P_n *be the vector space of all real polynomials with degrees at most* n*. Let* $D : P_n \to P_n$ *be given by differentiation:* $D(p) = p'$*. Let* π *be a real polynomial. What is the minimal polynomial of the transformation* $\pi(D)$*?*

Problem 19 *Prove or supply a counterexample: If* f *is a continuous complex valued function defined on a connected open subset of the complex plane and if* f^2 *is analytic, then* f *is analytic.*

Problem 20 *Let f be a C^2 function on the real line. Assume f is bounded with bounded second derivative. Let*

$$A = \sup_{x\in\mathbb{R}} |f(x)|, \quad B = \sup_{x\in\mathbb{R}} |f''(x)|.$$

Prove that

$$\sup_{x\in\mathbb{R}} |f'(x)| \leqslant 2\sqrt{AB}.$$