Preliminary Exam - Fall 1983

Problem 1 *Evaluate*

$$\int_0^\infty (\operatorname{sech} x)^2 \cos \lambda x \, dx$$

where λ *is a real constant and*

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}\cdot$$

Problem 2 *Let* $M_{n\times n}(\mathbf{F})$ *be the ring of* $n\times n$ *matrices over a field* $\mathbf{F}$*. For* $n \geqslant 1$ *does there exist a ring homomorphism from* $M_{(n+1)\times(n+1)}(\mathbf{F})$ *onto* $M_{n\times n}(\mathbf{F})$*?*

Problem 3 *Let* $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ *be a function which is continuously differentiable and whose partial derivatives are uniformly bounded:*

$$\left| \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) \right| \leqslant M$$

for all $(x_1, \ldots, x_n) \neq (0, \ldots, 0)$*. Show that if* $n \geqslant 2$*, then* f *can be extended to a continuous function defined on all of* $\mathbb{R}^n$*. Show that this is false if* $n = 1$ *by giving a counterexample.*

Problem 4 *Prove or disprove (by giving a counterexample), the following assertion: Every infinite sequence* $x_1, x_2, \ldots$ *of real numbers has either a nondecreasing subsequence or a nonincreasing subsequence.*

Problem 5 *Let* A *be the* $n \times n$ *matrix which has zeros on the main diagonal and ones everywhere else. Find the eigenvalues and eigenspaces of* A *and compute* $\det(A)$.

Problem 6 *Consider the polynomial*

$$p(z) = z^5 + z^3 + 5z^2 + 2.$$

How many zeros (counting multiplicities) does p *have in the annular region* $1 < |z| < 2$*?*

Problem 7 *Let G be a finite group and suppose that $G \times G$ has exactly four normal subgroups. Show that G is simple and nonabelian.*

Problem 8 *Let A be a linear transformation on $\mathbb{R}^3$ whose matrix (relative to the usual basis for $\mathbb{R}^3$) is both symmetric and orthogonal. Prove that A is either plus or minus the identity, or a rotation by $180°$ about some axis in $\mathbb{R}^3$, or a reflection about some two-dimensional subspace of $\mathbb{R}^3$.*

Problem 9 *For which real values of p does the differential equation*

$$y'' + 2py' + y = 3$$

admit solutions $y = f(x)$ with infinitely many critical points?

Problem 10 *Let $f : [0, \infty) \to \mathbb{R}$ be a uniformly continuous function with the property that*

$$\lim_{b \to \infty} \int_0^b f(x)\, dx$$

exists (as a finite limit). Show that

$$\lim_{x \to \infty} f(x) = 0.$$

Problem 11 *Prove or supply a counterexample: If f and g are C^1 real valued functions on $(0, 1)$, if*

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0,$$

if g and g' never vanish, and if

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = c,$$

then

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = c.$$

Problem 12 *Let $r_1, r_2, \ldots, r_n$ be distinct complex numbers. Show that a rational function of the form*

$$f(z) = \frac{b_0 + b_1 z + \cdots + b_{n-2} z^{n-2} + b_{n-1} z^{n-1}}{(z - r_1)(z - r_2) \cdots (z - r_n)}$$

can be written as a sum

$$f(z) = \frac{A_1}{z - r_1} + \frac{A_2}{z - r_2} + \cdots + \frac{A_n}{z - r_n}$$

for suitable constants $A_1, \ldots, A_n$.

Problem 13 1. *Let* $u(t)$ *be a real valued differentiable function of a real variable* t *which satisfies an inequality of the form*

$$u'(t) \leqslant au(t), \quad t \geqslant 0, \quad u(0) \leqslant b,$$

where a *and* b *are positive constants. Starting from first principles, derive an upper bound for* $u(t)$ *for* $t > 0$.

2. *Let* $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ *be a differentiable function from* $\mathbb{R}$ *to* $\mathbb{R}^n$ *which satisfies a differential equation of the form*

$$x'(t) = f(x(t)),$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous function. Assuming that* f *satisfies the condition*

$$\langle f(y), y \rangle \leqslant \|y\|^2, \quad y \in \mathbb{R}^n$$

derive an inequality showing that the norm $\|x(t)\|$ *grows, at most, exponentially.*

Problem 14 *Let* V *be a finite-dimensional complex vector space and let* A *and* B *be linear operators on* V *such that* $AB = BA$. *Prove that if* A *and* B *can each be diagonalized, then there is a basis for* V *which simultaneously diagonalizes* A *and* B.

Problem 15 1. *Let* f *be a complex function which is analytic on an open set containing the disc* $|z| \leqslant 1$, *and which is real valued on the unit circle. Prove that* f *is constant.*

2. *Find a nonconstant function which is analytic at every point of the complex plane except for a single point on the unit circle* $|z| = 1$, *and which is real valued at every other point of the unit circle.*

Problem 16 *Let* $F(t) = (f_{ij}(t))$ *be an* $n \times n$ *matrix of continuously differentiable functions* $f_{ij} : \mathbb{R} \to \mathbb{R}$*, and let*

$$u(t) = \operatorname{tr}\left(F(t)^3\right).$$

Show that u *is differentiable and*

$$u'(t) = 3 \operatorname{tr}\left(F(t)^2 F'(t)\right).$$

Problem 17 *Prove that every finite integral domain is a field.*

Problem 18 *Let* $f, g : \mathbb{R} \to \mathbb{R}$ *be smooth functions with* $f(0) = 0$ *and* $f'(0) \neq 0$*. Consider the equation* $f(x) = tg(x)$*,* $t \in \mathbb{R}$*.*

1. *Show that in a suitably small interval* $|t| < \delta$*, there is a unique continuous function* $x(t)$ *which solves the equation and satisfies* $x(0) = 0$*.*

2. *Derive the first order Taylor expansion of* $x(t)$ *about* $t = 0$*.*

Problem 19 *Prove that if* p *is a prime number, then the polynomial*

$$f(x) = x^{p-1} + x^{p-2} + \cdots + 1$$

is irreducible in $\mathbb{Q}[x]$*.*

Problem 20 *Let* m *and* n *be positive integers, with* $m < n$*. Let* $M_{m \times n}$ *be the space of linear transformations of* $\mathbb{R}^m$ *into* $\mathbb{R}^n$ *(considered as* $n \times m$ *matrices) and let* L *be the set of transformations in* $M_{m \times n}$ *which have rank* m*.*

1. *Show that* L *is an open subset of* $M_{m \times n}$*.*

2. *Show that there is a continuous function* $T : L \to M_{m \times n}$ *such that*$T(A)A = I_m$ *for all* A*, where* I_m *is the identity on* $\mathbb{R}^m$*.*