Problem 1 Evaluate
\[ \int_{0}^{\infty} (\text{sech} \, x)^2 \cos \lambda x \, dx \]
where \( \lambda \) is a real constant and
\[ \text{sech} \, x = \frac{2}{e^x + e^{-x}}. \]

Problem 2 Let \( M_{n \times n}(F) \) be the ring of \( n \times n \) matrices over a field \( F \). For \( n \geq 1 \) does there exist a ring homomorphism from \( M_{(n+1) \times (n+1)}(F) \) onto \( M_{n \times n}(F) \)?

Problem 3 Let \( f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) be a function which is continuously differentiable and whose partial derivatives are uniformly bounded:
\[ \left| \frac{\partial f}{\partial x_i} (x_1, \ldots, x_n) \right| \leq M \]
for all \( (x_1, \ldots, x_n) \neq (0, \ldots, 0) \). Show that if \( n \geq 2 \), then \( f \) can be extended to a continuous function defined on all of \( \mathbb{R}^n \). Show that this is false if \( n = 1 \) by giving a counterexample.

Problem 4 Prove or disprove (by giving a counterexample), the following assertion: Every infinite sequence \( x_1, x_2, \ldots \) of real numbers has either a nondecreasing subsequence or a nonincreasing subsequence.

Problem 5 Let \( A \) be the \( n \times n \) matrix which has zeros on the main diagonal and ones everywhere else. Find the eigenvalues and eigenspaces of \( A \) and compute \( \det(A) \).

Problem 6 Consider the polynomial
\[ p(z) = z^5 + z^3 + 5z^2 + 2. \]
How many zeros (counting multiplicities) does \( p \) have in the annular region \( 1 < |z| < 2 \)?
Problem 7 Let $G$ be a finite group and suppose that $G \times G$ has exactly four normal subgroups. Show that $G$ is simple and nonabelian.

Problem 8 Let $A$ be a linear transformation on $\mathbb{R}^3$ whose matrix (relative to the usual basis for $\mathbb{R}^3$) is both symmetric and orthogonal. Prove that $A$ is either plus or minus the identity, or a rotation by 180° about some axis in $\mathbb{R}^3$, or a reflection about some two-dimensional subspace of $\mathbb{R}^3$.

Problem 9 For which real values of $p$ does the differential equation
\[ y'' + 2py' + y = 3 \]
admit solutions $y = f(x)$ with infinitely many critical points?

Problem 10 Let $f : [0, \infty) \to \mathbb{R}$ be a uniformly continuous function with the property that
\[ \lim_{b \to \infty} \int_0^b f(x) \, dx \]
exists (as a finite limit). Show that
\[ \lim_{x \to \infty} f(x) = 0. \]

Problem 11 Prove or supply a counterexample: If $f$ and $g$ are $C^1$ real valued functions on $(0,1)$, if
\[ \lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0, \]
if $g$ and $g'$ never vanish, and if
\[ \lim_{x \to 0} \frac{f(x)}{g(x)} = c, \]
then
\[ \lim_{x \to 0} \frac{f'(x)}{g'(x)} = c. \]

Problem 12 Let $r_1, r_2, \ldots, r_n$ be distinct complex numbers. Show that a rational function of the form
\[ f(z) = \frac{b_0 + b_1 z + \cdots + b_{n-2} z^{n-2} + b_{n-1} z^{n-1}}{(z - r_1)(z - r_2)\cdots(z - r_n)} \]
can be written as a sum

\[ f(z) = \frac{A_1}{z - r_1} + \frac{A_2}{z - r_2} + \cdots + \frac{A_n}{z - r_n} \]

for suitable constants \( A_1, \ldots, A_n \).

**Problem 13**

1. Let \( u(t) \) be a real valued differentiable function of a real variable \( t \) which satisfies an inequality of the form

\[ u'(t) \leq au(t), \quad t \geq 0, \quad u(0) \leq b, \]

where \( a \) and \( b \) are positive constants. Starting from first principles, derive an upper bound for \( u(t) \) for \( t > 0 \).

2. Let \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) be a differentiable function from \( \mathbb{R} \) to \( \mathbb{R}^n \) which satisfies a differential equation of the form

\[ x'(t) = f(x(t)), \]

where \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a continuous function. Assuming that \( f \) satisfies the condition

\[ \langle f(y), y \rangle \leq \|y\|^2, \quad y \in \mathbb{R}^n \]

derive an inequality showing that the norm \( \|x(t)\| \) grows, at most, exponentially.

**Problem 14** Let \( V \) be a finite-dimensional complex vector space and let \( A \) and \( B \) be linear operators on \( V \) such that \( AB = BA \). Prove that if \( A \) and \( B \) can each be diagonalized, then there is a basis for \( V \) which simultaneously diagonalizes \( A \) and \( B \).

**Problem 15**

1. Let \( f \) be a complex function which is analytic on an open set containing the disc \( |z| \leq 1 \), and which is real valued on the unit circle. Prove that \( f \) is constant.

2. Find a nonconstant function which is analytic at every point of the complex plane except for a single point on the unit circle \( |z| = 1 \), and which is real valued at every other point of the unit circle.
Problem 16 Let \( F(t) = (f_{ij}(t)) \) be an \( n \times n \) matrix of continuously differentiable functions \( f_{ij} : \mathbb{R} \to \mathbb{R} \), and let

\[
    u(t) = \text{tr} \left( F(t)^3 \right).
\]

Show that \( u \) is differentiable and

\[
    u'(t) = 3 \text{tr} \left( F(t)^2 F'(t) \right).
\]

Problem 17 Prove that every finite integral domain is a field.

Problem 18 Let \( f, g : \mathbb{R} \to \mathbb{R} \) be smooth functions with \( f(0) = 0 \) and \( f'(0) \neq 0 \). Consider the equation \( f(x) = tg(x) \), \( t \in \mathbb{R} \).

1. Show that in a suitably small interval \( |t| < \delta \), there is a unique continuous function \( x(t) \) which solves the equation and satisfies \( x(0) = 0 \).

2. Derive the first order Taylor expansion of \( x(t) \) about \( t = 0 \).

Problem 19 Prove that if \( p \) is a prime number, then the polynomial

\[
    f(x) = x^{p-1} + x^{p-2} + \cdots + 1
\]

is irreducible in \( \mathbb{Q}[x] \).

Problem 20 Let \( m \) and \( n \) be positive integers, with \( m < n \). Let \( M_{m \times n} \) be the space of linear transformations of \( \mathbb{R}^m \) into \( \mathbb{R}^n \) (considered as \( n \times m \) matrices) and let \( L \) be the set of transformations in \( M_{m \times n} \) which have rank \( m \).

1. Show that \( L \) is an open subset of \( M_{m \times n} \).

2. Show that there is a continuous function \( T : L \to M_{m \times n} \) such that \( T(A)A = I_m \) for all \( A \), where \( I_m \) is the identity on \( \mathbb{R}^m \).