Preliminary Exam - Fall 1982

Problem 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous nowhere vanishing function, and consider the differential equation*

$$\frac{dy}{dx} = f(y).$$

1. *For each real number c, show that this equation has a unique, continuously differentiable solution $y = y(x)$ on a neighborhood of 0 which satisfies the initial condition $y(0) = c$.*

2. *Deduce the conditions on f under which the solution y exists for all $x \in \mathbb{R}$, for every initial value c.*

Problem 2 *Consider the polynomial ring $R = \mathbb{Z}[x]$ and the ideal $\mathfrak{I}$ of R generated by 7 and $x - 3$.*

1. *Show that for each $r \in R$, there is an integer α satisfying $0 \leqslant \alpha \leqslant 6$ such that $r - \alpha \in \mathfrak{I}$.*

2. *Find α in the special case $r = x^{250} + 15x^{14} + x^2 + 5$.*

Problem 3 *Let*

$$\cot(\pi z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

be the Laurent expansion for $\cot(\pi z)$ on the annulus $1 < |z| < 2$. Compute the a_n for $n < 0$.

Problem 4 *Let M be an $n \times n$ matrix of real numbers. Prove or disprove: The dimension of the subspace of $\mathbb{R}^n$ generated by the rows of M is equal to the dimension of the subspace of $\mathbb{R}^n$ generated by the columns of M.*

Problem 5 *Let $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ be nonnegative continuous functions on $[0, 1]$ such that the limit*

$$\lim_{n \to \infty} \int_0^1 x^k \varphi_n(x)\, dx$$

exists for every $k = 0, 1, \dots$. *Show that the limit*

$$\lim_{n\to\infty} \int_0^1 f(x)\varphi_n(x)\,dx$$

exists for every continuous function f *on* $[0, 1]$.

Problem 6 *Let* T *be a linear transformation on a finite-dimensional* $\mathbb{C}$-*vector space* V, *and let* f *be a polynomial with coefficients in* $\mathbb{C}$. *If* λ *is an eigenvalue of* T, *show that* $f(\lambda)$ *is an eigenvalue of* $f(T)$. *Is every eigenvalue of* $f(T)$ *necessarily obtained in this way?*

Problem 7 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{\cos \pi x}{4x^2 - 1}\,dx\,.$$

Problem 8 *Let*

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in \mathbb{R},\ a > 0 \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}.$$

1. *Show that* N *is a normal subgroup of* G *and prove that* G/N *is isomorphic to* $\mathbb{R}$.

2. *Find a normal subgroup* N' *of* G *satisfying* $N \subset N' \subset G$ *(where the inclusions are proper), or prove that there is no such subgroup.*

Problem 9 *Let* f *be a real valued continuous nonnegative function on* $[0, 1]$ *such that*

$$f(t)^2 \leqslant 1 + 2\int_0^t f(s)\,ds$$

for $t \in [0, 1]$. *Show that* $f(t) \leqslant 1 + t$ *for* $t \in [0, 1]$.

Problem 10 *Let* a *and* b *be complex numbers whose real parts are negative or* 0. *Prove the inequality* $|e^a - e^b| \leqslant |a - b|$.

Problem 11

1. *Prove that there is no continuous map from the closed interval* $[0, 1]$ *onto the open interval* $(0, 1)$.

2. *Find a continuous surjective map from the open interval $(0,1)$ onto the closed interval $[0,1]$.*

3. *Prove that no map in Part 2 can be bijective.*

Problem 12 *Let A and B be complex $n \times n$ matrices having the same rank. Suppose that $A^2 = A$ and $B^2 = B$. Prove that A and B are similar.*

Problem 13 *Let $f_1, f_2, \ldots$ be continuous functions on $[0,1]$ satisfying $f_1 \geqslant f_2 \geqslant \cdots$ and such that $\lim_{n\to\infty} f_n(x) = 0$ for each x. Must the sequence $\{f_n\}$ converge to 0 uniformly on $[0,1]$?*

Problem 14 *Let A be an $n \times n$ complex matrix, and let B be the Hermitian transpose of A (i.e., $b_{ij} = \overline{a}_{ji}$). Suppose that A and B commute with each other. Consider the linear transformations α and β on $\mathbb{C}^n$ defined by A and B. Prove that α and β have the same image and the same kernel.*

Problem 15 *Let A be a subgroup of an abelian group B. Assume that A is a direct summand of B, i.e., there exists a subgroup X of B such that $A \cap X = 0$ and such that $B = X + A$. Suppose that C is a subgroup of B and satisfying $A \subset C \subset B$. Is A necessarily a direct summand of C?*

Problem 16 *Find all pairs of C^∞ functions $x(t)$ and $y(t)$ on $\mathbb{R}$ satisfying*

$$x'(t) = 2x(t) - y(t), \qquad y'(t) = x(t).$$

Problem 17 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\, dx\,.$$

Problem 18 *Let K be a continuous function on the unit square$0 \leqslant x, y \leqslant 1$ satisfying $|K(x,y)| < 1$ for all x and y. Show that there is a continuous function $f(x)$ on $[0,1]$ such that we have*

$$f(x) + \int_0^1 K(x,y) f(y)\, dy = e^{x^2}\,.$$

Can there be more than one such function f?

Problem 19 *Let a and b be nonzero complex numbers and$f(z) = az + bz^{-1}$. Determine the image under f of the unit circle $\{z \mid |z| = 1\}$.*

Problem 20 *Let G be the abelian group given by generators x, y, and z and by the three relations:*

$$2x + 4y + 6z = 0,$$
$$8x + 4z = 4y,$$
$$6x = 8y + 2z.$$

Write G as a product of cyclic groups. How many elements of G have order 2?