Problem 1 Evaluate the integral
\[ \int_{-\infty}^{\infty} \frac{\cos x}{1 + x^4} \, dx. \]

Problem 2 Consider an autonomous system of differential equations
\[ \frac{dx_i}{dt} = F_i(x_1, \ldots, x_n), \]
where \( F = (F_1, \ldots, F_n) : \mathbb{R}^n \to \mathbb{R}^n \) is a \( C^1 \) vector field.

1. Let \( U \) and \( V \) be two solutions on \( a < t < b \). Assuming that
\[ \langle DF(x)z, z \rangle \leq 0 \]
for all \( x, z \) in \( \mathbb{R}^n \), show that \( \|U(t) - V(t)\|^2 \) is a decreasing function of \( t \).

2. Let \( W(t) \) be a solution defined for \( t > 0 \). Assuming that
\[ \langle DF(x)z, z \rangle \leq -\|z\|^2, \]
show that there exists \( C \in \mathbb{R}^n \) such that
\[ \lim_{t \to \infty} W(t) = C. \]

Problem 3 Let \( S_n \) be the group of all permutations of \( n \) objects and let \( G \) be a subgroup of \( S_n \) of order \( p^k \), where \( p \) is a prime not dividing \( n \). Show that \( G \) has a fixed point; that is, one of the objects is left fixed by every element of \( G \).

Problem 4 Prove the following three statements about real \( n \times n \) matrices.

1. If \( A \) is an orthogonal matrix whose eigenvalues are all different from \( -1 \), then \( I + A \) is nonsingular and \( S = (I - A)(I + A)^{-1} \) is skew-symmetric.
2. If $S$ is a skew-symmetric matrix, then $A = (I - S)(I + S)^{-1}$ is an orthogonal matrix with no eigenvalue equal to $-1$.

3. The correspondence $A \leftrightarrow S$ from Parts 1 and 2 is one-to-one.

**Problem 5** The Fibonacci numbers $f_1, f_2, \ldots$ are defined recursively by $f_1 = 1$, $f_2 = 2$, and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$. Show that

$$
\lim_{n \to \infty} \frac{f_{n+1}}{f_n}
$$

exists, and evaluate the limit.

*Note: See also Problem ??.*

**Problem 6** Let $f$ and $g$ be continuous functions on $\mathbb{R}$ such that $f(x + 1) = f(x)$, $g(x + 1) = g(x)$, for all $x \in \mathbb{R}$. Prove that

$$
\lim_{n \to \infty} \int_0^1 f(x) g(nx) \, dx = \int_0^1 f(x) \, dx \int_0^1 g(x) \, dx.
$$

**Problem 7** Find a specific polynomial with rational coefficients having $\sqrt{2} + \sqrt{3}$ as a root.

**Problem 8**

1. How many zeros does the function $f(z) = 3z^{100} - e^z$ have inside the unit circle (counting multiplicities)?

2. Are the zeros distinct?

**Problem 9** Let $M_{2 \times 2}$ be the vector space of all real $2 \times 2$ matrices. Let

$$
A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}
$$

and define a linear transformation $L : M_{2 \times 2} \to M_{2 \times 2}$ by $L(X) = AXB$. Compute the trace and the determinant of $L$.

**Problem 10** Let $A = (a_{ij})$ be an $n \times n$ matrix whose entries $a_{ij}$ are real valued differentiable functions defined on $\mathbb{R}$. Assume that the determinant $\det(A)$ of $A$ is everywhere positive. Let $B = (b_{ij})$ be the inverse matrix of $A$. Prove the formula

$$
\frac{d}{dt} \log (\det(A)) = \sum_{i,j=1}^n \frac{da_{ij}}{dt} b_{ji}.
$$
Problem 11 Consider the complex $3\times3$ matrix

$$A = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_2 & a_0 & a_1 \\ a_1 & a_2 & a_0 \end{pmatrix},$$

where $a_0, a_1, a_2 \in \mathbb{C}$.

1. Show that $A = a_0 I_3 + a_1 E + a_2 E^2$, where

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

2. Use Part 1 to find the complex eigenvalues of $A$.

3. Generalize Parts 1 and 2 to $n\times n$ matrices.

Problem 12 Let $a, b$ be real constants and let

$$u(x, y) = a^2 + b^2 + x^2 - y^2$$

Show that $u$ is harmonic and find an entire function $f(z)$ whose real part is $u$.

Correction: $u$ cannot be the real part of an entire function. Why? Change $u$ slightly and do the problem.

Problem 13 Let $f$ be a real valued function on $\mathbb{R}^n$ of class $C^2$. A point $x \in \mathbb{R}^n$ is a critical point of $f$ if all the partial derivatives of $f$ vanish at $x$; a critical point is nondegenerate if the $n \times n$ matrix

$$\left( \frac{\partial^2 f}{\partial x_i \partial x_j} (x) \right)$$

is nonsingular.

Let $x$ be a nondegenerate critical point of $f$. Prove that there is an open neighborhood of $x$ which contains no other critical points (i.e., the nondegenerate critical points are isolated).
Problem 14 Let $V : \mathbb{R}^n \to \mathbb{R}$ be a $C^1$ function and consider the system of second order differential equations

$$x_i''(t) = f_i(x(t)), \quad 1 \leq i \leq n,$$

where

$$f_i = -\frac{\partial V}{\partial x_i}.$$

Let $x(t) = (x_1(t), \ldots, x_n(t))$ be a solution of this system on a finite interval $a < t < b$.

1. Show that the function

$$H(t) = \frac{1}{2} \langle x'(t), x'(t) \rangle + V(x(t))$$

is constant for $a < t < b$.

2. Assuming that $V(x) \geq M > -\infty$ for all $x \in \mathbb{R}^n$, show that $x(t)$, $x'(t)$, and $x''(t)$ are bounded on $a < t < b$, and then prove all three limits

$$\lim_{t \to b} x(t), \lim_{t \to b} x'(t), \lim_{t \to b} x''(t)$$

exist.

Problem 15 Let $f$ be a holomorphic map of the unit disc $D = \{z \mid |z| < 1\}$ into itself, which is not the identity map $f(z) = z$. Show that $f$ can have, at most, one fixed point.

Problem 16 Let $G$ be a group with three normal subgroups $N_1, N_2, N_3$. Suppose $N_i \cap N_j = \{e\}$ and $N_iN_j = G$ for all $i, j$ with $i \neq j$. Show that $G$ is abelian and $N_i$ is isomorphic to $N_j$ for all $i, j$.

Problem 17 Let $f$ be a continuous function on $[0, 1]$. Evaluate the following limits.

1. 

$$\lim_{n \to \infty} \int_0^1 x^n f(x) \, dx.$$
2. 
\[ \lim_{n \to \infty} n \int_{0}^{1} x^n f(x) \, dx. \]

**Problem 18** Let \( A \) and \( B \) be two real \( n \times n \) matrices. Suppose there is a complex invertible \( n \times n \) matrix \( U \) such that \( A = UBU^{-1} \). Show that there is a real invertible \( n \times n \) matrix \( V \) such that \( A = VBV^{-1} \). (In other words, if two real matrices are similar over \( \mathbb{C} \), then they are similar over \( \mathbb{R} \).)

**Problem 19** Either prove or disprove (by a counterexample) each of the following statements:

1. Let \( f : \mathbb{R} \to \mathbb{R}, \ g : \mathbb{R} \to \mathbb{R} \) be such that 
   \[ \lim_{t \to a} g(t) = b \quad \text{and} \quad \lim_{t \to b} f(t) = c. \]
   Then 
   \[ \lim_{t \to a} f(g(t)) = c. \]

2. If \( f : \mathbb{R} \to \mathbb{R} \) is continuous and \( U \) is an open set in \( \mathbb{R} \), then \( f(U) \) is an open set in \( \mathbb{R} \).

3. Let \( f \) be of class \( C^\infty \) on the interval \((-1, 1)\). Suppose that \( |f^{(n)}(x)| \leq 1 \) for all \( n \geq 1 \) and all \( x \) in the interval. Then \( f \) is real analytic; that is, it has a convergent power series expansion in a neighborhood of each point of the interval.

**Problem 20** Let \( G \) be a group of order 10 which has a normal subgroup of order 2. Prove that \( G \) is abelian.