Preliminary Exam - Fall 1981

Problem 1 *Evaluate the integral*

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^4}\, dx\,.$$

Problem 2 *Consider an autonomous system of differential equations*

$$\frac{dx_i}{dt} = F_i(x_1, \ldots, x_n),$$

where $F = (F_1, \ldots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$ *is a* C^1 *vector field.*

1. *Let* U *and* V *be two solutions on* $a < t < b$*. Assuming that*

   $$\langle DF(x)z, z\rangle \leqslant 0$$

   for all x*,* z *in* $\mathbb{R}^n$*, show that* $\|U(t) - V(t)\|^2$ *is a decreasing function of* t*.*

2. *Let* $W(t)$ *be a solution defined for* $t > 0$*. Assuming that*

   $$\langle DF(x)z, z\rangle \leqslant -\|z\|^2,$$

   show that there exists $C \in \mathbb{R}^n$ *such that*

   $$\lim_{t\to\infty} W(t) = C.$$

Problem 3 *Let* S_n *be the group of all permutations of* n *objects and let* G *be a subgroup of* S_n *of order* p^k*, where* p *is a prime not dividing* n*. Show that* G *has a* fixed point*; that is, one of the objects is left fixed by every element of* G*.*

Problem 4 *Prove the following three statements about real* $n \times n$ *matrices.*

1. *If* A *is an orthogonal matrix whose eigenvalues are all different from* -1*, then* $I + A$ *is nonsingular and* $S = (I - A)(I + A)^{-1}$ *is skew-symmetric.*

2. *If S is a skew-symmetric matrix, then $A = (I - S)(I + S)^{-1}$ is an orthogonal matrix with no eigenvalue equal to -1.*

3. *The correspondence $A \leftrightarrow S$ from Parts 1 and 2 is one-to-one.*

Problem 5 *The Fibonacci numbers $f_1, f_2, \ldots$ are defined recursively by $f_1 = 1$, $f_2 = 2$, and $f_{n+1} = f_n + f_{n-1}$ for $n \geqslant 2$. Show that*

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n}$$

exists, and evaluate the limit.
Note: See also Problem **??**.

Problem 6 *Let f and g be continuous functions on $\mathbb{R}$ such that$f(x + 1) = f(x)$, $g(x + 1) = g(x)$, for all $x \in \mathbb{R}$. Prove that*

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)\,dx = \int_0^1 f(x)\,dx \int_0^1 g(x)\,dx\,.$$

Problem 7 *Find a specific polynomial with rational coefficients having $\sqrt{2}+\sqrt[3]{3}$ as a root.*

Problem 8 1. *How many zeros does the function $f(z) = 3z^{100} - e^z$ have inside the unit circle (counting multiplicities)?*

2. *Are the zeros distinct?*

Problem 9 *Let $M_{2\times 2}$ be the vector space of all real 2×2 matrices. Let*

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}$$

and define a linear transformation $L : M_{2\times 2} \to M_{2\times 2}$ by $L(X) = AXB$. Compute the trace and the determinant of L.

Problem 10 *Let $A = (a_{ij})$ be an $n \times n$ matrix whose entries a_{ij} are real valued differentiable functions defined on $\mathbb{R}$. Assume that the determinant $\det(A)$ of A is everywhere positive. Let $B = (b_{ij})$ be the inverse matrix of A. Prove the formula*

$$\frac{d}{dt} \log\left(\det(A)\right) = \sum_{i,j=1}^{n} \frac{da_{ij}}{dt} b_{ji}.$$

Problem 11 *Consider the complex 3×3 matrix*

$$A = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_2 & a_0 & a_1 \\ a_1 & a_2 & a_0 \end{pmatrix},$$

where $a_0, a_1, a_2 \in \mathbb{C}$.

1. *Show that* $A = a_0 I_3 + a_1 E + a_2 E^2$, *where*

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

2. *Use Part 1 to find the complex eigenvalues of A.*

3. *Generalize Parts 1 and 2 to n×n matrices.*

Problem 12 *Let a, b be real constants and let*

$$u(x, y) = \frac{a^2 + b^2 + x^2 - y^2}{(a - x)^2 + (b - y)^2}.$$

Show that u is harmonic and find an entire function f(z) whose real part is u.

Correction: u cannot be the real part of an entire function. Why? Change u slightly and do the problem.

Problem 13 *Let* f *be a real valued function on* $\mathbb{R}^n$ *of class* C^2. *A point* $x \in \mathbb{R}^n$ *is a critical point of f if all the partial derivatives of f vanish at x; a critical point is* nondegenerate *if the* $n \times n$ *matrix*

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right)$$

is nonsingular.

Let x be a nondegenerate critical point of f. Prove that there is an open neighborhood of x which contains no other critical points (i.e., the nondegenerate critical points are isolated).

Problem 14 *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function and consider the system of second order differential equations*

$$x_i''(t) = f_i\left(x(t)\right), \quad 1 \leqslant i \leqslant n,$$

where

$$f_i = -\frac{\partial V}{\partial x_i}.$$

Let $x(t) = (x_1(t), \ldots, x_n(t))$ be a solution of this system on a finite interval $a < t < b$.

1. *Show that the function*

$$H(t) = \frac{1}{2}\langle x'(t), x'(t)\rangle + V(x(t))$$

   is constant for $a < t < b$.

2. *Assuming that $V(x) \geqslant M > -\infty$ for all $x \in \mathbb{R}^n$, show that $x(t)$, $x'(t)$, and $x''(t)$ are bounded on $a < t < b$, and then prove all three limits*

$$\lim_{t\to b} x(t), \quad \lim_{t\to b} x'(t), \quad \lim_{t\to b} x''(t)$$

   exist.

Problem 15 *Let f be a holomorphic map of the unit disc $\mathbb{D} = \{z \mid |z| < 1\}$ into itself, which is not the identity map $f(z) = z$. Show that f can have, at most, one fixed point.*

Problem 16 *Let G be a group with three normal subgroups N_1, N_2, N_3. Suppose $N_i \cap N_j = \{e\}$ and $N_iN_j = G$ for all i, j with $i \neq j$. Show that G is abelian and N_i is isomorphic to N_j for all i, j.*

Problem 17 *Let f be a continuous function on $[0, 1]$. Evaluate the following limits.*

1.

$$\lim_{n\to\infty} \int_0^1 x^n f(x)\, dx\,.$$

2.

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)\, dx\,.$$

Problem 18 *Let A and B be two real $n \times n$ matrices. Suppose there is a complex invertible $n \times n$ matrix U such that$A = UBU^{-1}$. Show that there is a real invertible $n \times n$ matrix V such that $A = VBV^{-1}$. (In other words, if two real matrices are similar over $\mathbb{C}$, then they are similar over $\mathbb{R}$.)*

Problem 19 *Either prove or disprove (by a counterexample) each of the following statements:*

1. *Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ be such that*

   $$\lim_{t\to a} g(t) = b \;\; and \;\; \lim_{t\to b} f(t) = c.$$

   Then

   $$\lim_{t\to a} f\left(g(t)\right) = c.$$

2. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and U is an open set in $\mathbb{R}$, then $f(U)$ is an open set in $\mathbb{R}$.*

3. *Let f be of class C^∞ on the interval $(-1, 1)$. Suppose that $|f^{(n)}(x)| \leqslant 1$ for all $n \geqslant 1$ and all x in the interval. Then f is* real analytic*; that is, it has a convergent power series expansion in a neighborhood of each point of the interval.*

Problem 20 *Let G be a group of order 10 which has a normal subgroup of order 2. Prove that G is abelian.*