Preliminary Exam - Fall 1980

Problem 1 *Define*

$$F(x) = \int_{\sin x}^{\cos x} e^{(t^2+xt)}\, dt.$$

Compute $F'(0)$.

Problem 2 *Are the matrices give below similar ?*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix} \qquad \textit{and} \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Problem 3 *Do there exist functions* $f(z)$ *and* $g(z)$ *that are analytic at* $z = 0$ *and that satisfy*

1. $f(1/n) = f(-1/n) = 1/n^2$, $n = 1, 2, \ldots,$
2. $g(1/n) = g(-1/n) = 1/n^3$, $n = 1, 2, \ldots$?

Problem 4 *Let* G *be the group of orthogonal transformations of* $\mathbb{R}^3$ *to* $\mathbb{R}^3$ *with determinant* 1*. Let* $v \in \mathbb{R}^3$, $|v| = 1$, *and let* $H_v = \{T \in G \mid Tv = v\}$.

1. *Show that* H_v *is a subgroup of* G.
2. *Let* $S_v = \{T \in G \mid T$ *is a rotation of* $180°$ *about a line orthogonal to* $v\}$. *Show that* S_v *is a coset of* H_v *in* G.

Problem 5 *Evaluate*

$$\int_0^\infty \frac{x^{m-1}}{1+x^n}\, dx$$

where n *and* m *are positive integers and* $0 < m < n$.

Problem 6 *Let* $M_{2\times 2}$ *be the ring of real 2×2 matrices and* $S \subset M_{2\times 2}$ *the subring of matrices of the form*

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

1. *Exhibit an isomorphism between S and $\mathbb{C}$.*

2. *Prove that*
$$A = \begin{pmatrix} 0 & 3 \\ -4 & 1 \end{pmatrix}$$
lies in a subring isomorphic to S.

3. *Prove that there is an $X \in M_{2\times 2}$ such that $X^4 + 13X = A$.*

Problem 7 *Let g be 2π-periodic, continuous on $[-\pi, \pi]$ and have Fourier series*
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,.$$
Let f be 2π-periodic and satisfy the differential equation
$$f''(x) + kf(x) = g(x)$$
where $k \neq n^2, n = 1, 2, 3, \ldots$. Find the Fourier series of f and prove that it converges everywhere.

Problem 8 *Let $\mathbf{F}_2 = \{0, 1\}$ be the field with two elements. Let G be the group of invertible 2×2 matrices with entries in $\mathbf{F}_2$. Show that G is isomorphic to S_3, the group of permutations of three objects.*

Problem 9 *For a real 2×2 matrix*
$$X = \begin{pmatrix} x & y \\ z & t \end{pmatrix},$$
let $\|X\| = x^2 + y^2 + z^2 + t^2$, and define a metric by $d(X, Y) = \|X - Y\|$. Let $\Sigma = \{X \mid \det(X) = 0\}$. Let
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$
Find the minimum distance from A to Σ and exhibit an $S \in \Sigma$ that achieves this minimum.

Problem 10 *Show that there is an $\varepsilon > 0$ such that if A is any real 2×2 matrix satisfying $|a_{ij}| \leqslant \varepsilon$ for all entries a_{ij} of A, then there is a real 2×2 matrix X such that $X^2 + X^t = A$, where X^t is the transpose of X. Is X unique?*

Problem 11 *Let $f(z)$ be an analytic function defined for $|z| \leqslant 1$ and let*

$$u(x, y) = \Re f(z), \qquad z = x + iy.$$

Prove that

$$\int_C \frac{\partial u}{\partial y}\, dx - \frac{\partial u}{\partial x}\, dy = 0$$

where C is the unit circle, $x^2 + y^2 = 1$.

Problem 12 *Prove that any group of order 6 is isomorphic to either $\mathbb{Z}_6$ or S_3 (the group of permutations of three objects).*

Problem 13 *Let $f(x) = \frac{1}{4} + x - x^2$. For any real number x, define a sequence (x_n) by $x_0 = x$ and $x_{n+1} = f(x_n)$. If the sequence converges, let x_∞ denote the limit.*

1. *For $x = 0$, show that the sequence is bounded and nondecreasing and find $x_\infty = \lambda$.*
2. *Find all $y \in \mathbb{R}$ such that $y_\infty = \lambda$.*

Problem 14 *Exhibit a set of 2×2 real matrices with the following property: A matrix A is similar to exactly one matrix in S provided A is a 2×2 invertible matrix of integers with all the roots of its characteristic polynomial on the unit circle.*

Problem 15 *Consider the differential equation $x'' + x' + x^3 = 0$ and the function $f(x, x') = (x + x')^2 + (x')^2 + x^4$.*

1. *Show that f decreases along trajectories of the differential equation.*
2. *Show that if $x(t)$ is any solution, then $(x(t), x'(t))$ tends to $(0, 0)$ as $t \to \infty$.*

Problem 16 *Suppose that A and B are real matrices such that $A^t = A$,*

$$v^t A v \geqslant 0$$

for all $v \in \mathbb{R}^n$ and

$$AB + BA = 0.$$

Show that $AB = BA = 0$ and give an example where neither A nor B is zero.

Problem 17 *Let P_n be a sequence of real polynomials of degree $\leqslant D$, a fixed integer. Suppose that $P_n(x) \to 0$ pointwise for $0 \leqslant x \leqslant 1$. Prove that $P_n \to 0$ uniformly on $[0, 1]$.*

Problem 18 *Suppose that f is analytic inside and on the unit circle $|z| = 1$ and satisfies $|f(z)| < 1$ for $|z| = 1$. Show that the equation $f(z) = z^3$ has exactly three solutions (counting multiplicities) inside the unit circle.*

Problem 19 *Let X be a compact metric space and $f : X \to X$ an isometry. Show that $f(X) = X$.*

Problem 20 *Let R be a ring with multiplicative identity 1. Call $x \in R$ a* unit *if $xy = yx = 1$ for some $y \in R$. Let $G(R)$ denote the set of units.*

1. *Prove $G(R)$ is a multiplicative group.*
2. *Let R be the ring of complex numbers $a+bi$, where a and b are integers. Prove $G(R)$ is isomorphic to $\mathbb{Z}_4$ (the additive group of integers modulo 4).*