Preliminary Exam - Fall 1979

Problem 1 *Prove that the polynomial*

$$p(z) = z^{47} - z^{23} + 2z^{11} - z^5 + 4z^2 + 1$$

has at least one root in the disc $|z| < 1$.

Problem 2 *Suppose that* f *is analytic on the open upper half-plane and satisfies* $|f(z)| \leqslant 1$ *for all* z, $f(i) = 0$. *How large can* $|f(2i)|$ *be under these conditions?*

Problem 3 *Prove that every finite group of order* > 2 *has a nontrivial automorphism.*

Problem 4 *Consider the polynomial ring* $\mathbb{Z}[x]$ *and the ideal* $\mathfrak{I}$ *generated by* 7 *and* $x - 3$.

1. *Show that for each* $r \in \mathbb{Z}[x]$, *there is an integer* α *satisfying* $0 \leqslant \alpha \leqslant 6$ *such that* $r - \alpha \in \mathfrak{I}$.
2. *Find* α *in the special case* $r = x^{250} + 15x^{14} + x^2 + 5$.

Problem 5 *Let* A *be a real* skew-symmetric *matrix* ($A_{ij} = -A_{ji}$). *Prove that* A *has even rank.*

Problem 6 *Let* N *be a linear operator on an* n*-dimensional vector space,* $n > 1$, *such that* $N^n = 0$, $N^{n-1} \neq 0$. *Prove there is no operator* X *with* $X^2 = N$.

Problem 7 *Let* V *be the vector space of sequences* (a_n) *of complex numbers. The shift operator* $S : V \to V$ *is defined by*

$$S((a_1, a_2, a_3, \ldots)) = (a_2, a_3, a_4, \ldots).$$

1. *Find the eigenvectors of* S.

2. *Show that the subspace W consisting of the sequences (x_n) with$x_{n+2} = x_{n+1} + x_n$ is a two-dimensional, S-invariant subspace of V and exhibit an explicit basis for W.*

3. *Find an explicit formula for the n^{th} Fibonacci number f_n, where$f_2 = f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$ for $n \geqslant 1$.*

Note: See also Problem **??**.

Problem 8 *Prove that*

$$\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) = \log 2.$$

Problem 9 *Given that*

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}\ ,$$

find $f'(t)$ explicitly, where

$$f(t) = \int_{-\infty}^{\infty} e^{-tx^2}\,dx, \qquad t > 0.$$

Problem 10 *Solve the differential equations*

$$\begin{aligned}\frac{dx}{dt} &= -3x + 10y\\ \frac{dy}{dt} &= -3x + 8y\,.\end{aligned}$$

Problem 11 *An accurate map of California is spread out flat on a table in Evans Hall, in Berkeley. Prove that there is exactly one point on the map lying directly over the point it represents.*

Problem 12 *Consider the following properties of a function$f : \mathbb{R}^n \to \mathbb{R}$:*

1. *f is continuous.*

2. *The graph of f is connected in $\mathbb{R}^n \times \mathbb{R}$.*

Prove or disprove the implications 1 $\Rightarrow$ 2, 2 $\Rightarrow$ 1.

Problem 13 *Let* $\{P_n\}$ *be a sequence of real polynomials of degree* $\leqslant D$, *a fixed integer. Suppose that* $P_n(x) \to 0$ *pointwise for* $0 \leqslant x \leqslant 1$. *Prove that* $P_n \to 0$ *uniformly on* $[0,1]$.

Problem 14 *Let* $y = y(x)$ *be a solution of the differential equation* $y'' = -|y|$ *with* $-\infty < x < \infty$, $y(0) = 1$ *and* $y'(0) = 0$.

1. *Show that* y *is an even function.*

2. *Show that* y *has exactly one zero on the positive real axis.*

Problem 15 *Suppose* f *and* g *are entire functions with* $|f(z)| \leqslant |g(z)|$ *for all* z. *Prove that* $f(z) = cg(z)$ *for some constant* c.

Problem 16 *Prove that*

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \frac{\pi}{\sin \pi\alpha}.$$

What restrictions must be placed on α*?*

Problem 17 *Let* A *be an* $n \times n$ *complex matrix. Prove there is a unitary matrix* U *such that* $B = UAU^{-1}$ *is upper-triangular:* $B_{jk} = 0$ *for* $j > k$.

Problem 18 *Let* B *denote the matrix*

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

where a, b, *and* c *are real and* $|a|$, $|b|$, *and* $|c|$ *are distinct. Show that there are exactly four symmetric matrices of the form* BQ, *where* Q *is a real orthogonal matrix of determinant* 1.

Problem 19 *Let* $M_{n\times n}(\mathbf{F})$ *be the ring of* $n\times n$ *matrices over a field* $\mathbf{F}$. *Prove that it has no 2-sided ideals except* $M_{n\times n}(\mathbf{F})$ *and* $\{0\}$.

Problem 20 *Let* G *be the abelian group defined by generators* x, y, *and* z, *and relations*

$$\begin{aligned} 15x + 3y &= 0 \\ 3x + 7y + 4z &= 0 \\ 18x + 14y + 8z &= 0. \end{aligned}$$

1. *Express G as a direct product of two cyclic groups.*
2. *Express G as a direct product of cyclic groups of prime power order.*
3. *How many elements of G have order 2?*