# Preliminary Exam - Fall 1977

**Problem 1** *Let*

$$A = \begin{pmatrix} 7 & 15 \\ -2 & -4 \end{pmatrix}.$$

*Find a real matrix $B$ such that $B^{-1}AB$ is diagonal.*

**Problem 2** 1. *Using only the axioms for a field $\mathbf{F}$, prove that a system of $m$ homogeneous linear equations in $n$ unknowns with$m < n$ and coefficients in $\mathbf{F}$ has a nonzero solution.*

2. *Use Part 1 to show that if $V$ is a vector space over $\mathbf{F}$ which is spanned by a finite number of elements, then every maximal linearly independent subset of $V$ has the same number of elements.*

**Problem 3** *Let $T$ be an $n \times n$ complex matrix. Show that*

$$\lim_{k \to \infty} T^k = 0$$

*if and only if all the eigenvalues of $T$ have absolute value less than $1$.*

**Problem 4** *Let $P$ be a linear operator on a finite-dimensional vector space over a finite field. Show that if $P$ is invertible, then $P^n = I$ for some positive integer $n$.*

**Problem 5** 1. *Show that the set of all units in a ring with unity form a group under multiplication. (A* unit *is an element having a two-sided multiplicative inverse.)*

2. *In the ring $\mathbb{Z}_n$ of integers $\bmod n$, show that $k$ is a unit if and only if $k$ and $n$ are relatively prime.*

3. *Suppose $n = pq$, where $p$ and $q$ are primes. Prove that the number of units in $\mathbb{Z}_n$ is $(p-1)(q-1)$.*

**Problem 6** *Let* $u : \mathbb{R}^2 \to \mathbb{R}$ *be the function defined by* $u(x, y) = x^3 - 3xy^2$. *Show that* $u$ *is harmonic and find* $v : \mathbb{R}^2 \to \mathbb{R}$ *such that the function* $f : \mathbb{C} \to \mathbb{C}$ *defined by*

$$f(x + iy) = u(x, y) + iv(x, y)$$

*is analytic.*

**Problem 7** *Evaluate*

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^{2n}}$$

*where* $n$ *is a positive integer.*

**Problem 8** *Find all solutions of the differential equation*

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = \sin t$$

*subject to the condition* $x(0) = 1$ *and* $x'(0) = 0$.

**Problem 9** *Let* $f : [0, 1] \to \mathbb{R}$ *be continuously differentiable, with* $f(0) = 0$. *Prove that*

$$\sup_{0 \leqslant x \leqslant 1} |f(x)| \leqslant \sqrt{\int_0^1 (f'(x))^2 \, dx}.$$

**Problem 10** *Let* $f_n : \mathbb{R} \to \mathbb{R}$ *be differentiable for each* $n = 1, 2, \ldots$ *with* $|f_n'(x)| \leqslant 1$ *for all* $n$ *and* $x$. *Assume*

$$\lim_{n \to \infty} f_n(x) = g(x)$$

*for all* $x$. *Prove that* $g : \mathbb{R} \to \mathbb{R}$ *is continuous.*

**Problem 11** *Show that the differential equation* $x' = 3x^2$ *has no solution such that* $x(0) = 1$ *and* $x(t)$ *is defined for all real numbers* $t$.

**Problem 12** *Let* $X \subset \mathbb{R}$ *be a nonempty connected set of real numbers. If every element of* $X$ *is rational, prove* $X$ *has only one element.*

**Problem 13** *Consider the following four types of transformations:*

$$z \mapsto z + b, \quad z \mapsto 1/z, \quad z \mapsto kz \quad (\textit{where } k \neq 0),$$

$$z \mapsto \frac{az + b}{cz + d} \quad (\textit{where } ad - bc \neq 0).$$

*Here, $z$ is a variable complex number and the other letters denote constant complex numbers. Show that each transformation takes circles to either circles or straight lines.*

**Problem 14** *If $a$ and $b$ are complex numbers and $a \neq 0$, the set $a^b$ consists of those complex numbers $c$ having a logarithm of the form $b\alpha$, for some logarithm $\alpha$ of $a$. (That is, $e^{b\alpha} = c$ and $e^\alpha = a$ for some complex number $\alpha$.) Describe set $a^b$ when $a = 1$ and $b = 1/3 + i$.*

**Problem 15** *Let $f : \mathbb{R}^n \to \mathbb{R}$ have continuous partial derivatives and satisfy*

$$\left| \frac{\partial f}{\partial x_j}(x) \right| \leqslant K$$

*for all $x = (x_1, \ldots, x_n)$, $j = 1, \ldots, n$. Prove that*

$$|f(x) - f(y)| \leqslant \sqrt{n} K \|x - y\|$$

*(where $\|u\| = \sqrt{u_1^2 + \cdots + u_n^2}$ ).*

**Problem 16** *Let $E$ and $F$ be vector spaces (not assumed to be finite-dimensional). Let $S : E \to F$ be a linear transformation.*

1. *Prove $S(E)$ is a vector space.*
2. *Show $S$ has a kernel $\{0\}$ if and only if $S$ is injective (i.e., one-to-one).*
3. *Assume $S$ is injective; prove $S^{-1} : S(E) \to E$ is linear.*

**Problem 17** *Let $G$ be the set of 3×3 real matrices with zeros below the diagonal and ones on the diagonal.*

1. *Prove $G$ is a group under matrix multiplication.*
2. *Determine the center of $G$.*

**Problem 18** *Suppose the nonzero complex number $\alpha$ is a root of a polynomial of degree $n$ with rational coefficients. Prove that $1/\alpha$ is also a root of a polynomial of degree $n$ with rational coefficients.*

**Problem 19** *Let $M$ be a real $3 \times 3$ matrix such that $M^3 = I$, $M \neq I$.*

1. *What are the eigenvalues of $M$?*
2. *Give an example of such a matrix.*

**Problem 20** *Let $\mathbb{C}^3$ denote the set of ordered triples of complex numbers. Define a map $F : \mathbb{C}^3 \to \mathbb{C}^3$ by*

$$F(u, v, w) = (u + v + w, uv + vw + wu, uvw).$$

*Prove that $F$ is onto but not one-to-one.*