# Preliminary Exam - Fall 2000

**Problem 1** *Let $V$ be a finite-dimensional vector space, and let$f : V \to V$ be a linear transformation. Let $W$ denote the image of $f$. Prove that the restriction of $f$ to $W$, considered as an endomorphism of $W$, has the same trace as $f : V \to V$.*

**Problem 2** *Let $A$ be a subset of a compact metric space $(X, d)$. Assume that, for every continuous function $f : X \to \mathbb{R}$, the restriction of $f$ to $A$ attains a maximum on $A$. Prove that $A$ is compact.*

**Problem 3** *Suppose $\mathbf{K}$ is a field and $R$ is a nonzero $\mathbf{K}$-algebra generated by two elements $a$ and $b$ which satisfy $a^2 = b^2 = 0$ and $(a + b)^2 = 1$. Show $R$ is isomorphic to $M_2(\mathbf{K})$ (the algebra of $2 \times 2$ matrices over $K$).*

**Problem 4** *Evaluate the integral*

$$I = \frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{\sin 4z}$$

*where the direction of integration is counterclockwise.*

**Problem 5** *Let $f$ be a real-valued differentiable function on $(-1, 1)$ such that $f(x)/x^2$ has a finite limit as $x \to 0$. Does it follow that $f''(0)$ exists? Give a proof or a counterexample.*

**Problem 6** *Suppose $V$ is a vector space over a field $\mathbf{K}$. If $U$ and $W$ are subspaces, let $E(U, W)$ be the set of linear endomorphisms $F$ of $V$ over $\mathbf{K}$ with the property that the image of $FU$ in $V/W$ is finite dimensional. Show that $E(U, U)$ is a subring of the ring of endomorphisms of $V$ with two-sided ideals $E(V, U)$ and $E(U, 0)$.*

**Problem 7** *Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous with $f(0) = 0$. Prove: there exists a positive number $B$ such that $|f(x)| \leqslant 1 + B|x|$, for all $x$.*

**Problem 8** *Let* $\mathbf{F}_p$ *denote the field of* $p$ *elements (*$p$ *prime). Let* $n$ *be a positive integer. Prove that there is a transformation* $A \in GL_n(\mathbf{F}_p)$ *(the group of invertible linear transformations from* $(\mathbf{F}_p)^n$ *into itself) which, as a permutation of the nonzero vectors of* $(\mathbf{F}_p)^n$*, acts as a single cycle of length* $p^n - 1$.

**Problem 9** *Assume the nonconstant entire function* $f$ *takes real values on two intersecting lines in the complex plane. Prove that the measure of either angle formed by the lines is a rational multiple of* $\pi$.

**Problem 10** *Find all real numbers* $t$ *for which the quadratic form* $Q_t$ *on* $\mathbb{R}^3$*, defined by*

$$Q_t(x_1, x_2, x_3) = 2x_1^2 + x_2^2 + 3x_3^2 + 2tx_1x_2 + 2x_1x_3 \ ,$$

*is positive definite.*

**Problem 11** *Let* $f_n : \mathbb{R}^k \to \mathbb{R}^m$ *be continuous* $(n = 1, 2, \ldots)$. *Let* $K$ *be a compact subset of* $\mathbb{R}^k$. *Suppose* $f_n \to f$ *uniformly on* $K$. *Prove that* $S = f(K) \cup \bigcup_{n=1}^{\infty} f_n(K)$ *is compact.*

**Problem 12** *Show that for each positive integer* $k$ *there exists a positive integer* $N$ *such that there are at least* $k$ *nonisomorphic groups of order* $N$.

**Problem 13** *Let* $f(z)$ *be the rational function* $p(z)/q(z)$*, where* $p(z)$ *and* $q(z)$ *are nonzero polynomials with complex coefficients, such that the degree of* $p(z)$ *is less than the degree of* $q(z)$*, and such that* $q(z)$ *has no complex zeros with nonnegative imaginary part. Prove that if* $z_0$ *is a complex number with positive imaginary part, then*

$$f(z_0) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(t)}{t - z_0} \, dt.$$

**Problem 14** *Let* $T:\mathbb{R}^n \to \mathbb{R}^n$ *be a linear transformation, where* $n>1$. *Prove that there is a 2-dimensional subspace* $M \subseteq \mathbb{R}^n$ *such that* $T(M) \subseteq M$.

**Problem 15** *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is a nonconstant function such that* $f(x) \leqslant f(y)$ *whenever* $x \leqslant y$. *Prove that there exist* $a \in \mathbb{R}$ *and* $c > 0$ *such that* $f(a + x) - f(a - x) \geqslant cx$ *for all* $x \in [0, 1]$.

**Problem 16** *Let $G$ be a finite group of order $n$ with the property that for each divisor $d$ of $n$ there is at most one subgroup in $G$ of order $d$. Show $G$ is cyclic.*

**Problem 17** *Let $U$ be a connected and simply connected open subset of the complex plane, and let $f$ be a holomorphic function on $U$. Suppose $a$ is a point of $U$ such that the Taylor series of $f$ at $a$ converges on an open disc $D$ that intersects the complement of $U$. Does it follow that $f$ extends to a holomorphic function on $U \cup D$? Give a proof or a counterexample.*

**Problem 18** *Let $R$ be a ring with identity, having fewer than eight elements. Prove that $R$ is commutative.*