Department of Mathematics, University of California, Berkeley

YOUR 1 OR 2 DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part A — Fall Semester 2014

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1. Please write your 1- or 2-digit exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, or calculators may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

_______ , _______ , _______ , _______ , _______ , _______

GRADE COMPUTATION

1A. _______	1B. _______	Calculus
2A. _______	2B. _______	Real analysis
3A. _______	3B. _______	Real analysis
4A. _______	4B. _______	Complex analysis
5A. _______	5B. _______	Complex analysis
6A. _______	6B. _______	Linear algebra
7A. _______	7B. _______	Linear algebra
8A. _______	8B. _______	Abstract algebra
9A. _______	9B. _______	Abstract algebra

Part A Subtotal: _______ Part B Subtotal: _______ Grand Total: _______

YOUR EXAM NUMBER ___
Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

Let $a(n)$ be the number of ways that Harry Potter can buy a new broomstick valued at n knuts using bronze knuts, silver sickles worth 29 knuts, and gold galleons worth 17 sickles. Find $\sum_n a(n)z^n$ and $\lim_{n\to\infty} a(n)/n^2$.

Solution:

YOUR EXAM NUMBER ____

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Problem 2A. *Score:*

Suppose that f_n is a sequence of non-negative continuous functions on the unit interval. Find counterexamples to three of the following four inequalities:

$$\limsup_n \int_0^1 f_n(x)dx \leq \int_0^1 \limsup_n f_n(x)dx$$

$$\limsup_n \int_0^1 f_n(x)dx \geq \int_0^1 \limsup_n f_n(x)dx$$

$$\liminf_n \int_0^1 f_n(x)dx \leq \int_0^1 \liminf_n f_n(x)dx$$

$$\liminf_n \int_0^1 f_n(x)dx \geq \int_0^1 \liminf_n f_n(x)dx$$

(The remaining inequality always holds; you do not need to prove this.)

Solution:

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Problem 3A. *Score:*

Suppose that f is a twice-differentiable real-valued function on the real line such that $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ for all x. Find, with proof, a constant b such that $|f'(x)| < b$ for all x.

Solution:

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Problem 4A. *Score:*

Let D be the set consisting of the open unit disk together with the point 1. Show that the power series

$$\sum_{n>0} z^{3^n}/n - z^{2\times 3^n}/n$$

converges at all points of D. By examining points with argument of the form $\pi/3^k$, show that the function it converges to is not continuous.

Solution:

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Problem 5A. *Score:*

(a) Suppose that $P(z) = c(z - a_1) \cdots (z - a_n)$ is a complex polynomial. If z has positive real part and all the roots a_i have negative real part, show that $P'(z)/P(z)$ has positive real part.
(b) Show that all the roots of the derivative P' of a complex polynomial lie in the convex hull of the roots of P.

Solution:

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Problem 6A. *Score:*

Find the order of the group of linear transformations preserving a non-degenerate skew-symmetric bilinear form on a 4-dimensional vector space over the field with 3 elements, and find the number of such forms.

Solution:

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Problem 7A. *Score:*

Find a basis of the intersection of the subspace of $\mathbb{R}^4$ spanned by $(1,1,0,0)$, $(0,1,1,0)$, $(0,0,1,1)$ and the subspace spanned by $(1,0,t,0)$, $(0,1,0,t)$, where t is given.

Solution:

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Problem 8A. *Score:*

Let X be a totally ordered set, (i.e. equipped with a non-reflexive, transitive binary relation $<$ such that for every $x \neq y$, either $x < y$ or $y < x$). Let $L(X)$ denote the set of subsets $S \subseteq X$ with the property that for all $y \in S$ and $x < y$, $x \in S$. Find, with proof:

(a) a countably infinite totally ordered set X for which $L(X)$ has the smallest possible cardinality;

(b) a countably infinite totally ordered set X for which $L(X)$ has the largest possible cardinality.

Solution:

YOUR EXAM NUMBER ____

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Problem 9A. *Score:*

(a) By counting the number of pairs (g, x) with $g \in G, x \in X, g(x) = x$, show that the number of orbits of a finite group G acting on a finite set X is the average number of fixed points of elements of the group.

(b) In how many ways (up to symmetries of the hexagon) can one color the vertices of a regular hexagon using 4 colors?

Solution.

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GRADUATE PRELIMINARY EXAMINATION, Part B — Fall Semester 2014

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PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

YOUR EXAM NUMBER ____

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Problem 1B. *Score:*

Using induction or otherwise, show that the polynomial $P_n(x) = 1 + x^1/1! + ... + x^n/n!$ has exactly 1 real zero if n is odd and none if n is even.

Solution:

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Problem 2B. *Score:*

Either prove or describe a counterexample to the following statement: If a continuous real-valued function on the plane is bounded on all straight lines then it is bounded.

Solution:

YOUR EXAM NUMBER ___

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Problem 3B. *Score:*

Let $f : [0,1]\times[0,1] \to \mathbb{R}$ be continuous and assume that for all $x \in [0,1]$ there is a unique y_x such that $f(x, y_x) = \max\{f(x,y); y \in [0,1]\}$. Let $g(x) = y_x$. Show that $g : [0,1] \to [0,1]$ is continuous.

Solution:

YOUR EXAM NUMBER ____

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Problem 4B. *Score:*

Find four power series f_1, f_2, f_3, f_4 with radius of convergence 1 such that f_1, f_2 converge at 1 but f_3, f_4 do not, and the functions given by f_1, f_3 can be extended to functions holomorphic in a neighborhood of 1, but the functions given by f_2, f_4 cannot be.

Solution:

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Problem 5B. *Score:*

Let f be a doubly-periodic meromorphic function: $f(z+1) = f(z) = f(z+i)$ for all $z \in \mathbf{C}$. Let z_a be the zeroes of f inside the unit square $0 < Rez, Imz < 1$, w_b be its poles inside the square, and k_a and l_b be respective multiplicities. Assuming that f has no zeroes or poles on the boundary of the square, prove that

$$\sum_a k_a z_a - \sum_b l_b w_b \in \mathbb{Z}[i],$$

that is, is a Gaussian integer. *Hint:* Show that the following integral along the boundary of the square is a Gaussian integer:

$$\frac{1}{2\pi i} \oint z \frac{f'(z)}{f(z)} dz.$$

Solution:

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Problem 6B. *Score:*

Show that there is a sequence of 4×4 matrices $A(n)$ with real entries, which converges to

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and such that $A(n)$ has 4 distinct real eigenvalues, two of which are positive and two negative.

Solution:

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Problem 7B. *Score:*

Let n be a fixed positive integer, and define two n by n real symmetric matrices A and B to be equivalent if there is a non-singular real matrix C with $CAC^T = B$ (where C^T is the transpose of C). How many equivalence classes are there?

Solution:

YOUR EXAM NUMBER ____

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Problem 8B. *Score:*

Determine, up to isomorphism, all finite groups G such that G has exactly three conjugacy classes.

Solution:

YOUR EXAM NUMBER ___

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Problem 9B. *Score:*

How many roots does the polynomial $x^{100000}-1$ have in the finite field $\mathbb{F}_{65537}$? ($65537 = 2^{16}+1$ is a prime.)

Solution.