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Mathematics 128A
Final Examination
December 17, 1998

SHOW YOUR WORK COMPLETELY AND NEATLY. Total points = 140.

1. Give an example of an invertible $3 \times 3$ matrix $A$, a vector $b$, and an approximate solution of $Ax = b$ whose residual error is $\leq 10^{-4}$ but whose error is $\geq 1$. Justify your answer. (You may use any norm as long as you specify it.)

2. a) Describe briefly the strategy for deriving the Runge-Kutta methods for solving ODE's.

b) Define what is meant by the local truncation error, and the local order, for a single-step method for solving ODE's.

c) Derive a specific Runge-Kutta method of local order 3. Include a precise explanation of how you know that your method is of local order 3.

3. a) Define precisely what it means for a convergent sequence of numbers to converge linearly.

b) View $g(x) = 4x/(3x + 1)$ as an iteration function. Note that 1 is a fixed-point for $g$. Prove that for any initial guess which is $> 1$ the resulting iteration sequence will converge linearly to 1.

(over)
4. a) Derive the simple Simpson rule for numerical integration.

b) Derive the composite Simpson rule.

c) Very briefly discuss precisely the advantages of the composite Simpson rule compared to the composite trapezoid rule.

5. Find the LU decomposition of

\[ A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 1 & 4 & 1 \end{pmatrix} \]

Check your answer.

6. Find a positive integer, \( n \), such that if \( p \) is the polynomial which interpolates \( f(x) = \cos(x) \) at \( n \) equispaced points in the interval \([3, 5]\), then \(|f(x) - p(x)| < 10^{-4}\) for every \( x \) in that interval. Justify your answer.

7. Let \( p \) be a polynomial of degree \( n + 1 \) which, for the interval \([2, 6]\) and the weight function \( w(x) = 1 \), is orthogonal to all polynomials of lower degree. Assume that you have already proved that \( p \) then has \( n \) distinct roots in the interior of \([2, 6]\). Prove directly from this that the integration rule obtained by interpolating functions at the roots of \( p \) and integrating over \([2, 6]\) is exact for polynomials of degree \( \leq 2n + 1 \).