SHOW YOUR WORK COMPLETELY AND NEATLY. Total points = 140.

1. Let $p$ be the polynomial which interpolates $f(x) = \sqrt{x}$ at the points 4, 9, and 16. Find an upper bound for $|f(x) - p(x)|$ for $9 \leq x \leq 10$. Justify your answer.

2. Let $A = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$.
   a) Find the LU decomposition of $A$.
   b) Use the LU decomposition of $A$ to solve the equation $Ax = b$ for $b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
   c) Obtain an estimate for the condition number of $A$ for a norm of your choice (specify it). Justify your answer.

3. a) Describe briefly the strategy for deriving the Runge-Kutta methods for solving ODE's.
   b) Explain precisely the derivation, following your strategy above, of the second order Runge-Kutta method
      $y_{j+1} = y_j + h(f(x_j, y_j) + f(x_j + h, y_j + hf(x_j, y_j))/2$.
   c) Define what is meant by the local truncation error for a single-step method for solving ODE’s.
   d) Show that the local truncation error for the above method is of order $h^3$.
4. a) Give a brief but precise geometric explanation of how one obtains the formula for the Newton-Raphson method for finding the zeros of a function. (Include a precise statement of the formula).

b) Define precisely what it means for a convergent sequence of numbers to converge quadratically.

c) Show precisely why the Newton-Raphson method often gives quadratic convergence.

5. The Laguerre polynomials, \( L_n \), are orthogonal polynomials on \([0, +\infty)\) for the weight function \( w(x) = e^{-x} \). They satisfy the beautiful recursion relation

\[
(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x),
\]

while \( L_0(x) = 1 \) and \( L_1(x) = 1-x \). Derive the Gaussian two-point integration formula for

\[
\int_0^\infty f(x) e^{-x} \, dx.
\]

6. Suppose you are equipped with a pocket calculator with trig functions (but without an integrate key), and you need to find

\[
\int_0^\infty \cos(x^2) \, dx
\]

with an error of \(< 10^{-4}\). Explain precisely how you would proceed to do this, e.g. at what points you would evaluate \( \cos(x^2) \) and what you would do with the values. (You do not need to carry out the computation.) Prove that your procedure will give an answer of the required accuracy.