Math 16B, Final Exam, Fall 1996
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Part I. Shorter questions. Show work and put answers in boxes. 3 points each. No partial credit. No credit without work shown.

1. Find \( \frac{\partial}{\partial x} \left( \frac{\sin x + \cos y}{\sin x - \cos y} \right) \) and simplify.

2. Find \( \int x^2 e^{-x} dx \).

3. Find \( \int_{e}^{2e} \frac{dx}{x \ln x} \).

4. Find \( \int_{0}^{\infty} xe^{-x^2} dx \).
5. Use the fact that a circle of radius $r$ has area $A = \pi r^2$ to find the area of the ellipse $9x^2 + 25y^2 = 225$.

6. If $y' = 3t + ty$ and $y(0) = 5$, find $y = f(t)$.

7. If $y' = 3t + t^2$ and $y(0) = 5$, find $y = f(t)$.

8. Find the rational number, in lowest terms, whose decimal expansion is .027027027...

9. Find the sum of the infinite series $2 + \frac{4}{5} + \frac{8}{25} + \frac{16}{125} + \frac{32}{625} + \ldots$
10. Use a Taylor series to approximate the definite integral $\int_{0}^{0.1} e^{x^2} \, dx$
to ten decimal places.

Part II. Longer questions. 10 points each. Show your work and put answers in boxes. No
credit without work.

1. Let $f(x, y) = 2x^2 - x^4 - y^2$.
   (a) Find all points at which $f(x, y)$ has a potential relative maximum or minimum.

   (b) Use the second derivative test at each of the points found in part (a) above,
to determine whether the function has a relative maximum, a relative minimum,
neither of these, or no conclusion from the test.

2. Integrate
   (a) $\int \sin^3 x \, dx$. Hint: Use the identity $\sin^2 x + \cos^2 x = 1$.

   (b) $\int x \sec^2 x \, dx$. 

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3. Consider the differential equation $y' = y^2 - 3y - 4$.
   (a) Draw the graph of $z = y^2 - 3y - 4$ in the $yz$-plane.

   (b) Sketch solutions of the differential equations in the $ty$-plane, showing constant
   solutions and the solutions with initial conditions $y(0) = 0$ and $y(0) = 3$. Indicate
   where the solutions are concave up, or concave down, and mark any inflection
   points.

4. Find the first three nonzero terms of the Taylor series for $f(x) = \tan x$ around
   $x = 0$. (Be sure to write your answer in simplest form.)
5. Given the Taylor series \( \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \)

(a) Find the first five terms of the Taylor series for \( \ln(1 + 2x) \).

(b) Find the function (in simplest form) whose Taylor series is \( 2 + 3x + 4x^2 + 5x^3 + 6x^4 + \ldots \). Hint: compare to the derivative of the series for \( 1/(1-x) \).
6. The XYZ musical instrument company plans to make \( x \) xylophones and \( y \) yellow synthesizers. Because of restrictions on the time of the expert technicians and the raw materials needed, \( x \) and \( y \) must satisfy the equation \( 4x^2 + 25y^2 = 50,000 \). The company makes a profit of \$20\) for each xylophone and \$100\) for each yellow synthesizer.

(a) Find the production levels \( x \) and \( y \) which will maximize profits, and

(b) Find the resulting profit to the company.
7. Suppose that your parents set up a fund for your college education. They deposit $40,000 into a bank account on January 1st of your first year. (We will assume for simplicity that you start school in January.) This account earns interest, compounded continuously, at a rate of 6% per year. You have to make continuous withdrawals at the rate of $1,000 per month to pay for your tuition, room and board, etc. (We will also assume that your expenses are spread out evenly throughout the year.)

(a) Write a differential equation for \( y = \) the amount of money in the account at time \( t \) in years.

(b) Solve the equation to find the function \( y = f(t) \).

(c) Now answer this question: Will you be able to complete four years of college with this fund, or will you have to get a job to supplement your income? If the money will run out before the end of four years, in which month of which year will that happen?