Math 113  FINAL EXAM
May 13, 1996  Prof. Wu

1. (5%) Prove that for an integer n, 3 ∣ n ⇔ 3 ∣ (sum of digits of n).

2. (5%) Let f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 be a polynomial with integer coefficients, and let r be a rational number such that f(r) = 0. Show that r has to be an integer and r ∣ a_0.

3. (5%) Find a minimal polynomial of \sqrt{1 + \sqrt{3}} over Q. (Be sure to prove that it is minimal.)

4. (5%) Let n be a positive integer \geq 2 such that n ∣ (b^{n-1} - 1) for all integers b which are not a multiple of n. What can you say about n?

5. (5%) Do the nonzero elements of Z_{13} form a cyclic group under multiplication? Give reasons.

6. (10%) Let p be a prime.
   (a) Prove: p \bigg| \binom{p}{k} \text{ for } k = 1, \cdots, p - 1, \text{ where } \binom{p}{k} \equiv \frac{p!}{k!(p-k)!}.
   (b) Prove: the mapping f : Z_p \rightarrow Z_p defined by f(k) = k^p for all k ∈ Z_p is a field isomorphism.

7. (10%) Is x^4 + 2x + 3 irreducible over \mathbb{R}? Is it irreducible over Q? Give reasons.

8. (10%) Let F \equiv \{a + ib : a, b ∈ Q\} and let K \equiv Q[x]/(x^2 + 1)Q[x]. Show that F is isomorphic to K as fields by defining a map φ : F \rightarrow K and show that φ has all the requisite properties.

9. (10%) If β is a root of x^3 - x + 1, find some p(x) ∈ Q[x] so that (β^2 - 2)p(β) = 1.

10. (10%) Let ζ = e^{2πi/3}. Compute (Q(ζ, √5) : Q(ζ)).

11. (25%) In (a)-(d) below, each part could be done independently.
   (a) Assume that if p is a prime, then z^{p-1} + z^{p-2} + \cdots + 1 is irreducible over Q. Compute (Q(cos(2π/7) + i sin(2π/7)) : Q). (Each step should be clearly explained.)
   (b) Suppose the regular 7-gon can be constructed with straightedge and compass. Explain why (Q(cos(2π/7) : Q) = 2^k for some k ∈ Z^+.
   (c) If F \equiv Q(cos(2π/7)), show that (F(i sin(2π/7)) : F) = 1 or 2.
   (d) Use (b) and (c) to conclude that if the regular 7-gon can be constructed with straightedge and compass, then (Q(cos(2π/7) + i sin(2π/7)) : Q) = 2^m for some m ∈ Z^+.
   (e) What can you conclude from (a) and (d)? What is your guess concerning the construction of the regular 11-gon, the regular 13-gon, the regular 23-gon, etc.?