Math H1A Fall, 1995

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Final Examination—December 13, 1995

This is an "open book" exam. You may consult your notes and textbook. Grading is based on completeness, clarity, and accuracy. Please write in complete English sentences and explain your reasoning carefully. Time allotted: three hours.

1 *(3 points).* Evaluate $\lim_{x\to 3}\dfrac{x^2-6x+9}{x^2-4x+3}$.

2a *(4 points).* Find $f'(1)$ if $f(x)=\sqrt[3]{x}$.

2b *(4 points).* Evaluate $\int_{-1}^{2} x^2\sqrt{1+x^3}\,dx$.

2c *(4 points).* Differentiate $f(x)=x^{1/x}$.

2d *(4 points).* Calculate $\dfrac{d}{dt}\log(\sin(t))$.

3 *(5 points).* Working directly with the definition of "limit," prove that $\lim_{x\to 1/2}\dfrac{1}{x}=2$.

4a *(3 points).* Show that $\dfrac{1}{\sqrt{n+1}}\geq 2\left(\sqrt{n+2}-\sqrt{n+1}\right)$ for all integers $n\geq 1$. (It might help to multiply both sides by the positive quantity $\sqrt{n+2}+\sqrt{n+1}$.)

4b *(4 points).* Use mathematical induction and the result of part (a) to show:

$$(\star)\qquad 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}\geq 2(\sqrt{n+1}-1)$$

for $n\geq 1$.

4c *(2 points).* Show that $\int_1^{n+1}\dfrac{1}{\sqrt{x}}\,dx=2(\sqrt{n+1}-1)$. Does this suggest a geometric interpretation of $(\star)$?

5a *(4 points).* Show that $I(x):=\int_0^x\dfrac{1}{1+t^2}\,dt+\int_0^{1/x}\dfrac{1}{1+t^2}\,dt$ does not depend on x.

5b *(3 points).* Calculate $I(1)$.

6 *(5 points).* Suppose that $f(x)$ is a continuous function on $[0,1]$ for which $\int_0^1\left(f(x)\right)^2dx$ vanishes. Show that f is identically 0.

7 *(5 points).* Suppose that b is non-zero. Prove that $x^4+b^4=(x+b)^4$ only when $x=0$. (If $a^4+b^4=(a+b)^4$, apply Rolle's theorem to $f(x):=x^4+b^4-(x+b)^4$ on the interval of numbers between 0 and a.)

8 *(4 points).* Let $A=\{\,x\,|\,x^2\leq 6 \text{ and } x \text{ is rational}\,\}$. Decide whether $\sup A$ exists. If $\sup A$ exists, is it an element of A?