Math 54, Section 1: Differential equations and linear algebra.
Fall 1994, H. W. Lenstra, Jr.
Final examination, December 12, 1994.

Name:
Section number:
T.A.:

List of discussion sections:
101  S. Simic
102  A. Gottlieb
103  G. Anderson
104  G. Anderson
105  S. Simic
106  T. Walker
107  A. Gottlieb
108  L. Pyle
109  L. Pyle

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Problem 1. (20 points)
Solve the system of differential equations

\[ x_1'(t) = 3x_1(t) + 3x_2(t), \]
\[ x_2'(t) = -2x_1(t) - 4x_2(t) \]

with initial conditions

\[ x_1(0) = 1, \quad x_2(0) = 3. \]

Show your work.

Solution:
Problem 2. (20 points)

(a) Find three functions $y_1(x), y_2(x), y_3(x)$ defined on $(-\infty, \infty)$ whose Wronskian is given by

$$W(y_1, y_2, y_3)(x) = e^{4x}.$$ 

(b) Are the functions $y_1, y_2, y_3$ that you found linearly independent on $(-\infty, \infty)$? Justify your answer.

Solution:
Problem 3. (20 points)
Find a homogeneous third-order linear differential equation with constant coefficients that has

\[ y(x) = 3 \cdot e^{-x} - \cos(2x) \]

as a solution. Explain how you found it.

What is the general solution of that differential equation?

Solution:
Problem 4. (20 points)
For the function $f(x) = e^{x/\pi}$, draw a careful sketch of the graphs of the functions to which the following Fourier series converge on the interval $[0, 4\pi]$:
(a) the Fourier sine series of $f$ on $[0, \pi]$;
(b) the Fourier cosine series of $f$ on $[0, \pi]$;
(c) the ordinary Fourier series of $f$ on $[-\pi, \pi]$.
Pay particular attention to the discontinuities of the functions.
Note: you are not asked to compute the coefficients of those Fourier series.

Solution:
Problem 5. (20 points)
Find a function $u = u(x, t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, satisfying the partial differential equation
\begin{equation}
(*) \quad u_{xx} = -u_{tt} \quad (0 < x < \pi, \quad t > 0),
\end{equation}
with boundary conditions
\begin{equation*}
u(0, t) = u(\pi, t) = 0 \quad (t > 0)
\end{equation*}
and initial conditions
\begin{equation*}
u(x, 0) = \sin(3x), \quad \nu_t(x, 0) = 0 \quad (0 < x < \pi).
\end{equation*}

*Hint:* look for a function of the form $u(x, t) = X(x)T(t)$, as in sec. 10.6 of the textbook; but note that $(*)$ is *not* a wave equation, because of the minus sign.

**Solution:**