Math 1A

First Midterm Exam — September 23, 1993
80 minutes

1a (5 points). Express the derivative \( \frac{d}{dx} \left\{ x \sin\left( \frac{1}{x^2} \right) \right\} \bigg|_{x=23} \) as a limit. Do not evaluate the limit.

1b (5 points). Use the definition of the derivative to calculate \( f'(1) \) when \( f(x) = \sqrt{x} \). The formula \( b^3 - a^3 = (b - a)(b^2 + ab + a^2) \) may be helpful.

2. In the problems on this page, you may use the differentiation formulas we have derived in class:

a (5 points). Find the equation of the line tangent to the curve \( y = \frac{x^2}{x^3 + 1} \) at the point \((1, \frac{1}{2})\).

b (5 points). A sugar cube tumbles from a 98-meter tall campanile. How fast is it falling after \( t \) seconds? [If you use the formula \( s(t) = 4.9t^2 \), explain in a sentence or two what it means.]

c (2 points). How fast is the cube falling when it hits ground?

3 (8 points). What is the domain of the function \( g(t) = \sqrt{\frac{(t - 1)(t - 5)}{(t - 3)(t - 5)}} \)? Find the horizontal and vertical asymptotes of the curve \( u = g(t) \). Make a crude sketch of this curve, showing the asymptotes.

4a (5 points). Find \( \lim_{t \to -5^-} \left( \left\lfloor \frac{[t]}{|-t|} \right\rfloor \right) \), where \( [\ ] \) is the “greatest integer” function.

4b (5 points). Find \( \lim_{t \to -\infty} \left( \sqrt{t^2 - t + 2} - \sqrt{t^2 + t + 1} \right) \).

5 (5 points each).

a. Evaluate \( \lim_{b \to 2} \frac{b^{691} - 2^{691}}{b - 2} \) by using rules of differentiation, first expressing the limit as a derivative.

b. Suppose that \( f(\pi/2) = 12 \), \( f'(\pi/2) = 3 \). Using the methods discussed in class, calculate \( \lim_{\theta \to \pi/2} \frac{\cos(\theta)}{f(\theta) - 12} \).
6 (5 points each).

a. Find $f'(x)$ if $f(x) = \frac{1}{x^2} \sin(x)$.

b. Find $\lim_{x \to \infty} \frac{\sin(x)}{x^2}$. Explain your reasoning.

Second Midterm — October 28, 1993
80 minutes

1a (5 points). A sample of chalk contains 0.3 grams of radioactive dwininium, which has a half-life of 18 years. How many years must expire before the sample contains only 0.01 grams of radioactive dwininium?

1b (4 points). Find all possible values of $\cosh x$, given that $\sinh x = 5/12$.

2a (5 points). Use differentials to find an approximate value for $\sin 1^\circ$.

2b (5 points). Let $g$ be the function inverse to $f$. Calculate $g'(2)$ from the table

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>$f'(x)$</td>
<td>$\frac{1}{2}$</td>
<td>-1</td>
<td>0.2</td>
<td>-$\frac{1}{3}$</td>
<td>9</td>
</tr>
</tbody>
</table>

3. Find the derivative $\frac{dy}{dx}$ (each part is worth four points):

a. $y = \arcsin(\sqrt{x})$

b. $y = e^{x^2+1}$

c. $y = \log_x(\cos x)$ ($0 < x < \pi/2$).

4 (8 points). A slug and an ant left the base of the Campanile at midnight last night. The ant began moving directly north, toward Evans Hall. The slug moved east, toward a slugfest in Birge Hall. At 8AM this morning, the ant had traveled 60 feet and was moving north at 10 feet/hour. The slug was 80 feet east of the Campanile, but had started moving west at the rate of 5 feet/hour. At what rate was the distance between the slug and the ant changing at 8AM?

5 Evaluate the limits (four points each):

a. $\lim_{x \to 0^+} \frac{x}{x^2 + 125}$

b. $\lim_{n \to \infty} \left(1 + \frac{\ln 2}{n}\right)^n$
c. \( \lim_{t \to 0^-} \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) \)

6a (4 points). Find \( \frac{dy}{dx} \) if \( y = x^y \).

6b (3 points). Find the derivative \( y' \), given \( y^2 + 6xy + x^2 = 7 \).

6c (2 points). Find a formula for \( y'' \) in terms of \( x \), \( y \) and \( y' \) if \( y^2 + 6xy + x^2 = 7 \).

Final Exam — December 15, 1993

1a (5 points). Differentiate with respect to \( x \): \( \sqrt{x + \sqrt{x}} \).

1b (5 points). Find \( L(-4) \) given that \( L(-1) = 1 \) and that \( L'(x) = \frac{1}{x} \).

2a (6 points). Find \( \lim_{x \to 0} f(x)^{g(x)} \), where \( f(x) = (0.4)^{x^2} \) and \( g(x) = 3x^2 \).

2b (5 points). Evaluate \( \lim_{x \to 1^+} \frac{\ln|x|}{|2 - x - x^2|} \).

3a (5 points). Evaluate \( \int \frac{dx}{x[1 + (\ln x)^2]} \).

3b (6 points). Evaluate \( \int_{\pi/3}^{\pi/2} \frac{\cos x}{\sin x} dx \), simplifying your answer as much as possible.

4 (9 points). A rain gutter is to be constructed from a metal sheet of width 30cm by bending up one-third of the sheet on each side through an angle \( \theta \). How should \( \theta \) be chosen so that the gutter will carry the maximum amount of water? (A crude hand-drawn diagram was supplied.)

5a (6 points). Evaluate \( \lim_{n \to \infty} \frac{3}{n} \sum_{k=1}^{n} \cos \left( \frac{n + 3k}{n} \right) \).

5b (5 points). Let \( c \) be a real number. Show that the equation \( x^5 - 6x + c = 0 \) has at most one root in the interval \([-1, 1]\).

6a (6 points). Find all numbers \( a \) such that the line tangent to \( y = x^2 + 1 \) at the point \((a, a^2 + 1)\) passes through \((0, -8)\).

6b (5 points). Find the derivative of \( (\sin x)^{\tan x} \) with respect to \( x \).

Suppose that \( \mathcal{R} \) is the region lying between the graphs of \( y = x^3 \) and \( y = 27x \) in the part of the plane where \( x \) and \( y \) are positive.

7a (5 points). Find a definite integral which represents the area of \( \mathcal{R} \).

7b (6 points). Find a definite integral which represents the volume of the figure generated by revolving \( \mathcal{R} \) about the \( y \)-axis.
8a (6 points). Find \( \frac{d}{dx} \int_{\sin x}^{\cos x} \sqrt{t^3 + 1} \, dt \).

8b (4 points). Find the average value of \( \sin x \) on the interval \([0, \pi]\).

9a (5 points). Evaluate \( \lim_{h \to 0} \frac{\sinh \left( \frac{x}{2} + h \right) - \sinh \left( \frac{x}{2} - h \right)}{h} \).

9b (5 points). Find \( \frac{dy}{dx} \) at the point \((3, 1)\) on the curve \(2(x^2 + y^2)^2 = 25(x^2 - y^2)\).

10a (2 points). Bob and Jill lift a 90-pound Stairmaster over a distance of 30 feet. How much work do they perform?

10b (4 points). At 7PM, a large pizza is taken from a 415°F oven to a 65°F dining room. At 7:08PM, the pizza has cooled to 135°F. What is the temperature of the piece which remains at 7:16PM? (Assume the validity of Newton's law of cooling — the pizza cools at a rate proportional to the difference of its temperature and that of the room.)