Problem 1A. *Score:*

Show that

$$\int_0^1 x^{-x}\,\mathrm{d}x = \sum_{n=1}^{\infty} n^{-n}$$

Solution:
Write $x^{-x} = \mathrm{e}^{-x\log x}$, Taylor expand the exponential, and integrate term by term.

Problem 2A. *Score:*

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and satisfies $f'(x) > f(x)$ for all real x. Show that if $f(0) = 0$ then $f(x) > 0$ for all $x > 0$.

Solution:
Since $f'(0) > 0$ we have $f(x) = x \cdot f'(0) + o(|x|)$ in a neighborhood of zero, so there is a $t > 0$ such that f is positive on $(0, t)$. Assume for contradiction that $f(x) \leq 0$ for some $x > 0$ and let x_0 be the first such x. Then $f(x) > 0$ on $(0, x_0)$, which means that $f'(x) > 0$ on $(0, x_0)$, so $f(x_0) > f(0) = 0$, a contradiction

Problem 3A. *Score:*

Let X be a metric space.
(a) If U is a subset of X show that there is a unique open set $\neg U$ disjoint from U and containing all open sets disjoint from U.
(b) Give an example of an open set U with $U \neq \neg\neg U$
(c) Prove that for all open sets U, $\neg U = \neg\neg\neg U$. (Hint: if $A \subseteq B$ and $B \subseteq A$ then $A = B$.)

Solution: (a) Take $\neg U$ to be the union of all open sets disjoint from U, which is open as the union of any collection of open sets is open.

(b) Take X to be the real line and U to be the nonzero reals. Then $\neg U$ is empty so $\neg\neg U$ is the real line.

(c) We have $A \subseteq \neg\neg A$ and applying this to $A = \neg U$ we get $\neg U \subseteq \neg\neg\neg U$. On the other hand, if $A \subseteq B$ then $\neg B \subseteq \neg A$, and applying this to $A = U, B = \neg\neg U$ we get $\neg\neg\neg U \subseteq \neg U$.

Combining these gives $\neg\neg\neg U = \neg U$.

Problem 4A. *Score:*

Let a be a real number with $|a| < 1$. Prove that

$$\sum_{k=1}^{\infty} a^k \cos(k\theta) = \frac{-a^2 + a\cos\theta}{1 + a^2 - 2a\cos\theta}$$

Solution: We use the fact that for any complex number $z = e^{i\theta} = \cos\theta + i\sin\theta \in \mathbb{C}$

$$\frac{1}{1-az} = \sum_{k=0}^{\infty} a^k z^k = \sum_{k=0}^{\infty} a^k e^{ik\theta} = 1 + \sum_{k=1}^{\infty} a^k(\cos(k\theta) + i\sin(k\theta)).$$

Therefore

$$\sum_{k=1}^{\infty} a^k \cos(k\theta) = \Re\left(\frac{1}{1-az} - 1\right) = \Re\left(\frac{az}{1-az}\right) = \Re\left(\frac{az(1-a\bar{z})}{|1-az|^2}\right)$$
$$= \Re\left(\frac{a\bar{z} - a^2}{(1-a\cos\theta)^2 + (a\sin\theta)^2}\right) = \frac{a\cos\theta - a^2}{1 + a^2 - 2a\cos\theta}$$

Problem 5A. *Score:*

Describe a conformal map from the set

$$\{|z - 4i| < 4\} \cap \{|z - i| > 1\}$$

oto the open unit disk.

Solution: Compose

$$f_1 : z \to 1/z$$
$$f_2 : z \to 8\pi(z + i/2)/3$$
$$f_3 : z \to exp(z)$$
$$f_4 : z \to (z - i)/(z + i)$$

Problem 6A. *Score:*

Let A be an $n \times n$ matrix with real entries such that $(A-I)^m = 0$ for some $m \geq 1$. Prove that there exists an $n \times n$ matrix B with real entries such that $B^2 = A$.

Solution: Write $A = I + N$, so $N^m = 0$. Let $P(x)$ be the m-th Taylor polynomial of the function $\sqrt{1+x}$, so $P(x)^2 \equiv 1 + x \pmod{x^m}$. In other words

$$P(x)^2 = 1 + x + x^m Q(x)$$

for some $Q(x) \in \mathbb{R}[x]$. Then

$$P(N)^2 = I + N + N^m Q(N) = I + N = A,$$

so $B := P(N)$ satisfies $B^2 = A$. S

Problem 7A. *Score:*

Suppose $A = (a_{ij})$ is a real symmetric $n \times n$ matrix with nonnegative eigenvalues. Show that

$$|a_{ij}| \leq \sqrt{a_{ii} a_{jj}}$$

for all distinct $i, j \leq n$.

Solution:
Since A is symmetric with nonnegative eigenvalues, we may diagonalize A as $A = UDU^T$ with positive D, so $A = B^T B$ for $B^T = UD^{1/2}$. Thus, A is a Gram matrix, i.e., $a_{ij} = \langle v_i, v_j \rangle$ where v_i are the columns of B, so by Cauchy Schwartz $a_{ij} \leq \|v_i\| \|v_j\| \leq \sqrt{a_{ii} a_{jj}}$, as desired.

Problem 8A. *Score:*

For three non-zero integers a, b and c show that

$$\gcd(a, \operatorname{lcm}(b, c)) = \operatorname{lcm}(\gcd(a, b), \gcd(a, c)).$$

where gcd and lcm stand for the greatest common divisor and the least common multiple of two integers, respectively.

Solution: Given a prime p, let α, β, and γ be the exponents of p in the prime factorization of a, b, and c, respectively. Then it will suffice to show that

$$\min\{\alpha, \max\{\beta, \gamma\}\} = \max\{\min\{\alpha, \beta\}, \min\{\alpha, \gamma\}\} .$$

Without loss of generality, we may assume that $\beta \le \gamma$; in that case $\max\{\beta, \gamma\} = \gamma$ and $\min\{\alpha, \beta\} \le \min\{\alpha, \gamma\}$. Therefore the above equation is true because both sides are equal to $\min\{\alpha, \gamma\}$.

Problem 9A. *Score:*

Suppose a prime number p divides the order of a finite group G. Prove the existence of an element $g \in G$ of order p.

Solution: Consider the set $X = \{(g_1, \ldots, g_p) \in G^p \mid g_1 \cdots g_p = e\}$. It is acted upon by the cyclic group $\mathbf{Z}/p\mathbf{Z}$ with $1 \in \mathbf{Z}/p\mathbf{Z}$ acting as the cyclic shift

$$(g_1, \ldots, g_p) \longmapsto (g_p, g_1, \ldots, g_{p-1}).$$

A fixed point of this action is a constant p-tuple $(g, \ldots, g)$ such that $g^p = e$. The number of fixed points is not zero, since $(e, \ldots, e)$ is a fixed point, and is congruent modulo p to

$$|X| = |G|^{p-1},$$

i.e., it is divisible by p, since $p > 1$. It follows that there is an element $g \neq e$ with $g^p = e$.

Problem 1B. *Score:*

A mathematician (stupidly) tries to estimate $\pi^2/6 = \sum_{n=1}^{\infty} 1/n^2$ by taking the sum of the first N terms of the series. What is the smallest value of N such that the error of this approximation is at most 10^{-6}? Hint: integral test.

Solution:

The integral test shows that $1/(N+1) < \sum_{n=N+1}^{\infty} 1/n^2 < 1/N$, so $N = 10^6$.

Problem 2B. *Score:*

Suppose $p(z)$ is a nonconstant real polynomial such that for some real number a, $p(a) \neq 0$ and $p'(a) = p''(a) = 0$. Prove that p must have at least one nonreal zero.

Solution: Observe that if $q(z)$ is a real-rooted polynomial with distinct roots, then by Rolle's theorem $q'(z)$ is also real-rooted (since it has degree one less than the degree of q) and has the property that between every two roots of q' there is a root of q. Since polynomials with distinct roots are dense in the set of real-rooted polynomials, this implies that if q is any real-rooted polynomial and $q'(z)$ has a double root at z then $q(z) = 0$.

For the given polynomial $p'(z)$ has a double root at a, but $p(a) \neq 0$, so p cannot be real-rooted.

Problem 3B. *Score:*

Prove that a continuous function from $\mathbb{R}$ to $\mathbb{R}$ which maps open sets to open sets must be monotone.

Solution: We prove the contrapositive. Assume f is not monotone, i.e., there exist $a < b < c$ with $f(a) < f(b)$ and $f(b) > f(c)$ or with $f(a) > f(b)$ and $f(b) < f(c)$. In the first case, let m be the point at which $f(x)$ is maximized in $[a, c]$; such a point exists since f is continuous. Moreover we must have $m \neq a, c$ by the hypothesis. But now the image of (a, c) under f contains m, but does not contain a neighborhood of m, so f cannot map open sets to open sets.

The second case is completely analogous.

Problem 4B. *Score:*

Evaluate

$$\int_{-\infty}^{\infty} \frac{x - \sin x}{x^3} \, \mathrm{d}x.$$

Solution:

Integrate by parts twice to reduce to $(1/6) \int_{-\infty}^{\infty} \frac{\sin(x)}{x} \, \mathrm{d}x$, which is a standard example in complex analysis.

Problem 5B. *Score:*

Suppose $h(z)$ is entire, $h(0) = 3 + 4i$, and $|h(z)| \leq 5$ whenever $|z| < 1$. What is $h'(0)$?

Solution: We have $|h(0)| = \sqrt{9 + 16} = 5$, so $|h(0)| \geq |h(z)|$ for $z \in D = \{|z| < 1\}$. By the maximum modulus principle this is only possible if $h(z)$ is constant on D, which implies that $h'(0) = 0$.

Problem 6B. *Score:*

Show that if A is an $n \times n$ complex matrix satisfying

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

for all $i \in \{1, \ldots, n\}$, then A must be invertible.

Solution:
Assume $Ax = 0$ and choose i such that $|x_i| = \max_j |x_j|$. Then

$$|a_{ii}||x_i| \leq \sum_{j \neq i} |a_{ij}||x_j| \leq \sum_{j \neq i} |a_{ij}||x_i|$$

so that

$$\left(|a_{ii}| - \sum_{j \neq i} |a_{ij}| \right) |x_i| \leq 0.$$

Since the first factor is positive by assumption and the second is nonnegative, we must have $x_i = 0$. By choice of i we must have $x = 0$ so A is invertible.

Problem 7B. *Score:*

For a real symmetric positive definite matrix A and a vector $v \in R^n$, show that

$$\int_{\mathbb{R}^n} \exp(-x^T Ax + 2v^T x)\, \mathrm{d}x = \frac{\pi^{n/2}}{\sqrt{\det A}} \exp(v^T A^{-1} v)$$

You may assume that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

Solution:
Complete the square, orthogonally diagonalize A, change variables, and integrate.

Problem 8B. *Score:*

Show that there are no natural numbers $x, y \geq 1$ such that

$$x^2 + y^2 = 7xy.$$

Solution: Assume that there was such a solution. Taking remainders modulo 7 gives us

$$x^2 + y^2 \equiv 0 \mod 7.$$

The quadratic remainders modulo 7 are $0, 1, 2, 4$. The only two quadratic remainders whose sum is $\equiv 0$ are 0 and 0. So

$$x^2 \equiv y^2 \equiv 0 \mod 7.$$

It follows that x, y are both divisible by 7, i.e. $x = 7x_1$, $y = 7y_1$, for some natural numbers x_1, y_1. It follows that

$$x_1^2 + y_1^2 = 7x_1y_1.$$

Repeating this process would produce an infinite sequence of pairs $(x, y), (x_1, y_1), (x_2, y_2), \ldots$ such that x_i and y_i are strictly decreasing sequences of integers. Contradiction.

Problem 9B. *Score:*

Find the smallest n for which the permutation group S_n contains a cyclic subgroup of order 111.

Solution: Let the partition $n = n_1 + n_2 + ... + n_k$ represent the cycle structure of an element $g \in S_n$, *i.e.* g is a products of commuting cycles of the lengths $n_1 \leq n_2 \leq ... \leq n_k$. The order of the cyclic subgroup generated by g is obviously equal to the least common multiple of $n_1, ..., n_k$. We want this least common multiple to be $111 = 3 \cdot 37$. One of the possibilities is $(n_1, n_2, ..., n_k) = (3, 37)$ in which case $n = 3 + 37 = 40$. We claim that this value of n is the minimal possible. Indeed, if 111 is the least common multiple of $n_1, ..., n_k$ then each of the prime factors $3, 37$ divides at least one of the numbers n_i and moreover, the sum of such factors dividing n_i does not exceed their product and thus does not exceed n_i. This implies $n = n_1 + ... + n_k \geq 3 + 37 = 40$.