Department of Mathematics, University of California, Berkeley

YOUR 1 OR 2 DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part A — Fall Semester 2018

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1. Please write your 1- or 2-digit exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, calculators or electronic devices may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

GRADE COMPUTATION (for use by grader—do not write below)

1A. ______	1B. ______	Calculus
2A. ______	2B. ______	Real analysis
3A. ______	3B. ______	Real analysis
4A. ______	4B. ______	Complex analysis
5A. ______	5B. ______	Complex analysis
6A. ______	6B. ______	Linear algebra
7A. ______	7B. ______	Linear algebra
8A. ______	8B. ______	Abstract algebra
9A. ______	9B. ______	Abstract algebra

Part A Subtotal: ______ Part B Subtotal: ______ Grand Total: ______

YOUR EXAM NUMBER ____
Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

Show that

$$\int_0^1 x^{-x}\,\mathrm{d}x = \sum_{n=1}^{\infty} n^{-n}$$

Solution:

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Problem 2A. *Score:*

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and satisfies $f'(x) > f(x)$ for all real x. Show that if $f(0) = 0$ then $f(x) > 0$ for all $x > 0$.

Solution:

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Problem 3A. *Score:*

Let X be a metric space.

(a) If U is a subset of X show that there is a unique open set $\neg U$ disjoint from U and containing all open sets disjoint from U.

(b) Give an example of an open set U with $U \neq \neg\neg U$

(c) Prove that for all open sets U, $\neg U = \neg\neg\neg U$. (Hint: if $A \subseteq B$ and $B \subseteq A$ then $A = B$.)

Solution:

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Problem 4A. *Score:*

Let a be a real number with $|a| < 1$. Prove that

$$\sum_{k=1}^{\infty} a^k \cos(k\theta) = \frac{-a^2 + a\cos\theta}{1 + a^2 - 2a\cos\theta}$$

Solution:

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Problem 5A. *Score:*

Describe a conformal map from the set

$$\{|z-4i|<4\}\cap\{|z-i|>1\}$$

oto the open unit disk.

Solution:

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Problem 6A. *Score:*

Let A be an $n \times n$ matrix with real entries such that $(A - I)^m = 0$ for some $m \geq 1$. Prove that there exists an $n \times n$ matrix B with real entries such that $B^2 = A$.

Solution:

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Problem 7A. *Score:*

Suppose $A = (a_{ij})$ is a real symmetric $n \times n$ matrix with nonnegative eigenvalues. Show that

$$|a_{ij}| \leq \sqrt{a_{ii}a_{jj}}$$

for all distinct $i, j \leq n$.

Solution:

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Problem 8A. *Score:*

For three non-zero integers a, b and c show that

$$\gcd(a, \operatorname{lcm}(b, c)) = \operatorname{lcm}(\gcd(a, b), \gcd(a, c)).$$

where gcd and lcm stand for the greatest common divisor and the least common multiple of two integers, respectively.

Solution:

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Problem 9A. *Score:*

Suppose a prime number p divides the order of a finite group G. Prove the existence of an element $g \in G$ of order p.

Solution:

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GRADUATE PRELIMINARY EXAMINATION, Part B Fall Semester 2018

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PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

YOUR EXAM NUMBER ____
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Problem 1B. *Score:*

A mathematician (stupidly) tries to estimate $\pi^2/6 = \sum_{n=1}^{\infty} 1/n^2$ by taking the sum of the first N terms of the series. What is the smallest value of N such that the error of this approximation is at most 10^{-6}? Hint: integral test.

Solution:

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Problem 2B. *Score:*

Suppose $p(z)$ is a nonconstant real polynomial such that for some real number a, $p(a) \neq 0$ and $p'(a) = p''(a) = 0$. Prove that p must have at least one nonreal zero.

Solution:

YOUR EXAM NUMBER ___

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Problem 3B. *Score:*

Prove that a continuous function from $\mathbb{R}$ to $\mathbb{R}$ which maps open sets to open sets must be monotone.

Solution:

YOUR EXAM NUMBER ____
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Problem 4B. *Score:*

Evaluate
$$\int_{-\infty}^{\infty} \frac{x - \sin x}{x^3} \, \mathrm{d}x.$$

Solution:

YOUR EXAM NUMBER ___

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Problem 5B. *Score:*

Suppose $h(z)$ is entire, $h(0) = 3 + 4i$, and $|h(z)| \leq 5$ whenever $|z| < 1$. What is $h'(0)$?

Solution:

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Problem 6B. *Score:*

Show that if A is an $n \times n$ complex matrix satisfying

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

for all $i \in \{1, \ldots, n\}$, then A must be invertible.

Solution:

YOUR EXAM NUMBER ____

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Problem 7B. *Score:*

For a real symmetric positive definite matrix A and a vector $v \in R^n$, show that

$$\int_{\mathbb{R}^n} \exp(-x^T Ax + 2v^T x)\, \mathrm{d}x = \frac{\pi^{n/2}}{\sqrt{\det A}} \exp(v^T A^{-1} v)$$

You may assume that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

Solution:

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Problem 8B. *Score:*

Show that there are no natural numbers $x, y \geq 1$ such that

$$x^2 + y^2 = 7xy.$$

Solution:

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Problem 9B. *Score:*

Find the smallest n for which the permutation group S_n contains a cyclic subgroup of order 111.

Solution: