Department of Mathematics, University of California, Berkeley

YOUR 1 OR 2 DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part A — Fall Semester 2016

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1. Please write your 1- or 2-digit exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, calculators or electronic devices may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

GRADE COMPUTATION (for use by grader—do not write below)

1A. ______	1B. ______	Calculus
2A. ______	2B. ______	Real analysis
3A. ______	3B. ______	Real analysis
4A. ______	4B. ______	Complex analysis
5A. ______	5B. ______	Complex analysis
6A. ______	6B. ______	Linear algebra
7A. ______	7B. ______	Linear algebra
8A. ______	8B. ______	Abstract algebra
9A. ______	9B. ______	Abstract algebra

Part A Subtotal: ______ Part B Subtotal: ______ Grand Total: ______

YOUR EXAM NUMBER ____
Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

(a) Prove that if $s > 1$ then $\sum_{n>0} n^{-s} = \prod_p 1/(1-p^{-s})$, where the product is over all primes p.
(b) Prove that the sum $\sum_p 1/p$ over all primes p diverges.

Solution:

YOUR EXAM NUMBER ____
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Problem 2A. *Score:*

Let $x\colon [a,b] \to \mathbb{R}$ and $f\colon [a,b] \to \mathbb{R}$ be non-negative continuous functions satisfying

$$x^2(t) \le 1 + \int_a^t f(s)x(s)ds$$

for $a \le t \le b$. Show that

$$x(t) \le 1 + \frac{1}{2}\int_a^t f(s)ds$$

for $a \le t \le b$.

Solution:

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Problem 3A. *Score:*

Given $K \geq 0$, let Lip_K be the set of functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$.

(a) Show that the formula

$$d(f_1, f_2) = \sum_{j=1}^{\infty} 2^{-j} \sup_{z \in [-j,j]} |f_1(z) - f_2(z)|$$

converges and defines a metric d on Lip_K.

(b) Show that Lip_K is a complete metric space with this metric.

Solution:

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Problem 4A. *Score:*

Find

$$\int_{-\infty}^{\infty} \frac{\sin^3(x)}{x^3}\, dx.$$

Solution:

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Problem 5A. *Score:*

Is there a function $f(z)$ analytic in $\mathbb{C} \setminus \{0\}$ such that $|f(z)| \geq \frac{1}{\sqrt{|z|}}$ for all $z \neq 0$?

Solution:

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Problem 6A. *Score:*

Fix $N \geq 1$. Let $s_1, \ldots, s_N, t_1, \ldots, t_N$ be $2N$ complex numbers of magnitude less than or equal to 1. Let A be the $N \times N$ matrix with entries

$$A_{ij} = \exp(t_i s_j).$$

Show that for every $m \geq 1$ there is an $N \times N$ matrix B with rank less than or equal to m such that

$$|A_{ij} - B_{ij}| \leq \frac{2}{m!}$$

for all i and j.

Solution:

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Problem 7A. *Score:*

Let A and B be two $n \times n$ matrices with coefficients in $\mathbb{Q}$. For any field extension K of $\mathbb{Q}$, we say that A and B are similar over K if $A = PBP^{-1}$ for some $n \times n$ invertible matrix P with coefficients in K. Prove that A and B are similar over $\mathbb{Q}$ if and only if they are similar over $\mathbb{C}$.

Solution:

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Problem 8A. *Score:*

Let $M_2(\mathbb{Q})$ be the ring of all 2×2 matrices with coefficients in $\mathbb{Q}$. Describe all field extensions K of $\mathbb{Q}$ such that there is an injective ring homomorphism $K \to M_2(\mathbb{Q})$. (Note: we take the convention that a ring homomorphism maps the multiplicative identity to the multiplicative identity.)

Solution:

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Problem 9A. *Score:*

Let p be a prime number, $\mathbb{F}_p$ be the finite field of p elements, and $\mathrm{GL}_n(\mathbb{F}_p)$ be the finite group of all invertible $n \times n$ matrices with coefficients in $\mathbb{F}_p$. Find the order of $\mathrm{GL}_n(\mathbb{F}_p)$.

Solution:

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GRADUATE PRELIMINARY EXAMINATION, Part B Fall Semester 2016

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PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

YOUR EXAM NUMBER ___

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Problem 1B. *Score:*

Let $C = \int_{-\infty}^{\infty} e^{-x^2} dx$ and let S_n be the $(n-1)$-dimensional "surface area" of the unit sphere in R^n (so $S_2 = 2\pi$, $S_3 = 4\pi/3$).

(a) Prove that $C^n = S_n \Gamma(n/2)/2$, where $\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt$. (Evaluate the integral of $e^{-(x_1^2+\cdots+x_n^2)}$ over R^n in rectangular and polar coordinates.)

(b) Show that $s\,\Gamma(s) = \Gamma(s+1)$, $\Gamma(1) = 1$.

(c) Evaluate C. (Hint: $S_2 = 2\pi$.)

(d) Evaluate S_4.

Solution:

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Problem 2B. *Score:*

Let K be a compact subset of $\mathbb{R}^n$ and $f(x) = d(x, K)$ be the Euclidean distance from x to the nearest point of K.

(a) Show that f is continuous and $f(x) = 0$ if $x \in K$.

(b) Let $g(x) = \max(1 - f(x), 0)$. Show that $\int g^m$ converges to the n-dimensional volume of K as $m \to \infty$.

(The n-dimensional volume of K is defined to be $\int 1_K$, if the integral exists, where $1_K(x) = 1$ for $x \in K$, and $1_K(x) = 0$ for $x \notin K$.)

Solution:

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Problem 3B. *Score:*

(a) Suppose that I is a closed interval and f is a smooth function from I to I such that $|f'|$ is bounded by some number $r < 1$ on I. Let a_0 be in I and put $a_{n+1} = f(a_n)$. Prove that the sequence a_n tends to the unique root of $f(x) = x$ in I.

(b) Show that if a_0 is real and $a_{n+1} = \cos(a_n)$ then a_n tends to a root of $\cos(x) = x$.

Solution:

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Problem 4B. *Score:*

Put $f(z) = z(e^z - 1)$. Prove there exists an analytic function $h(z)$ defined near $z = 0$ such that $f(z) = h(z)^2$. Find the first 3 terms in the power series expansion $h(z) = \sum a_n z^n$. Does $h(z)$ extend to an entire function on $\mathbb{C}$?

Solution:

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Problem 5B. *Score:*

Let $f_t(z)$ be a family of entire functions depending analytically on $t \in \Delta$, where Δ is the open unit disk in $\mathbb{C}$. Suppose that for all t, $f_t(z)$ is non-vanishing on the unit circle S^1 in $\mathbb{C}$. Prove that for each $k \geq 0$,

$$N_k(t) = \sum_{|z|<1 : f_t(z)=0} z^k$$

is an analytic function of t (the zeroes of $f_t(z)$ are taken with multiplicity in the sum).

Solution:

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Problem 6B. *Score:*

Let A be an $m \times n$ matrix of rank r and B a $p \times q$ matrix of rank s. Find the dimension of the vector space of $n \times p$ matrices X such that $AXB = 0$.

Solution:

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Problem 7B. *Score:*

Find an example of a vector space V over the real numbers $\mathbb{R}$ and two linear maps $f, g : V \to V$ such that f is injective but not surjective and g is surjective but not injective and such that $f + g$ is equal to the identity map 1_V.

Hint: construct V as a subspace of the space of sequences of real numbers, closed under the linear maps

$$f(a_1, a_2, a_3, \ldots) = (a_1 - a_2, a_2 - a_3, \ldots)$$

and

$$g(a_1, a_2, a_3, \ldots) = (a_2, a_3, \ldots).$$

Solution:

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Problem 8B. *Score:*

Let G be a group and n be a positive integer. Assume that there exists a surjective group homomorphism $\mathbb{Z}^n \to G$ and an injective group homomorphism $\mathbb{Z}^n \to G$. Prove that the group G is isomorphic to $\mathbb{Z}^n$.

Solution:

YOUR EXAM NUMBER ___

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Problem 9B. *Score:*

Find (with proof) the number of groups of order 12 up to isomorphism. You may assume the Sylow theorems (if a prime power p^n is the largest power of p dividing the order of a group, then the group has subgroups of order p^n and the number of them is 1 mod p.)

Solution: