Department of Mathematics, University of California, Berkeley

STUDENT EXAM NUMBER ________

GRADUATE PRELIMINARY EXAMINATION, Part A — Fall Semester 2012

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1. Please write your 1- or 2-digit student exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, or calculators may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

____ , ____ , ____ , ____ , ____ , ____

GRADE COMPUTATION

1A. ____	1B. ____	Calculus
2A. ____	2B. ____	Real analysis
3A. ____	3B. ____	Real analysis
4A. ____	4B. ____	Complex analysis
5A. ____	5B. ____	Complex analysis
6A. ____	6B. ____	Linear algebra
7A. ____	7B. ____	Linear algebra
8A. ____	8B. ____	Abstract algebra
9A. ____	9B. ____	Abstract algebra

Part A Subtotal: ____ Part B Subtotal: ____ Grand Total: ____

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Please cross out this problem if you do not wish it graded

Problem 1A. Calculus *Score:*

Find the length of the spiral given in polar coordinates by $r = e^{\theta}$, $-\infty < \theta \leq 0$.

Solution:

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Please cross out this problem if you do not wish it graded

Problem 2A. Real analysis *Score:*

Prove or disprove the following assertion:

> If $f\colon \mathbb{R} \to \mathbb{R}$ has the property that $f([a,b])$ is a bounded closed interval for every $a \leq b$, then f is continuous.

Solution:

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Problem 3A. Real analysis *Score:*

Prove the existence of the limit

$$\lim_{n\to\infty} \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n.$$

Solution:

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Problem 4A. Complex analysis *Score:*

(a) Find the poles and residues of $1/(z^3\cos(z))$.

(b) Show that the integral of the function above over a square contour centered at the origin with side $2\pi N$ tends to zero as the integer N tends to infinity.

(c) Find the sum $1/1^3 - 1/3^3 + 1/5^3 - 1/7^3 + \cdots$.

Solution:

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Problem 5A. Complex analysis *Score:*

(a) Show that if $|z| < 1$ then there is a holomorphic function defined on some neighborhood of the unit disk whose only zero is at z and that has absolute value 1 on the unit circle.

(b) Suppose that f is a holomorphic function on the complex plane and is not identically zero. Show that there is a holomorphic function g defined in some open set containing the unit disk such that $|f(z)| = |g(z)|$ whenever $|z| = 1$, and such that g has no zeros in the open unit disk.

Solution:

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Problem 6A. Linear algebra *Score:*

If U, V, W are subspaces of a vector space such that any two have intersection zero, prove that

$$\dim(U+V+W) + \dim U + \dim V + \dim W \leq \dim(U+V) + \dim(V+W) + \dim(W+U)$$

and give an example where equality does not hold.

Solution:

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Problem 7A. Linear algebra *Score:*

Let I_n denote the $n \times n$ identity matrix, and J_n the $n \times n$ matrix with all entries equal to 1. Determine for which real numbers a the matrix $I_n + aJ_n$ is invertible, and find its inverse.

Solution:

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Problem 8A. Abstract algebra *Score:*

Let G be a finite Abelian group of order n. Suppose m is a square-free (not divisible by the square of a prime), positive integer dividing n. Show that G contains an element of order m. Give an example to show that this need not be true if m is not assumed to be square-free.

Solution:

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Problem 9A. Abstract algebra *Score:*

Does there exists a homomorphism of commutative rings with unit from $\mathbb{Z}[x]/(x^2+3)$ to $\mathbb{Z}[x]/(x^2-x+1)$? Either exhibit such a homomorphism, or prove that none exists.

Solution:

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GRADUATE PRELIMINARY EXAMINATION, Part B — Fall Semester 2012

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PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

STUDENT EXAM NUMBER ______________________

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Problem 1B. Calculus *Score:*

Prove that

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

In 1706 John Machin used this formula to calculate π to 100 decimal places. Explain briefly why he did not use the simpler formula $\frac{\pi}{4} = \arctan 1$.

Solution:

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Problem 2B. Real analysis *Score:*

(a) Find the sum $1 - 1/2 + 1/3 - 1/4 + \cdots$

(b) Find the sum $1 - 1/2 - 1/4 + 1/3 - 1/6 - 1/8 + 1/5 - 1/10 - 1/12 + \cdots$.

Solution:

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Problem 3B. Real analysis *Score:*

Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose that a map $\pi : X \to Y$ is a *submetry*; this means that for every $x \in X$ and any $r > 0$, the image of the closed r-ball around x is the closed r-ball around $\pi(x)$.

(a) Show that π is surjective if X is nonempty.

(b) Show that π is continuous.

(c) Show that π is open (meaning that the image of any open subset is open).

Solution:

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Problem 4B. Complex analysis *Score:*

Compute

$$\int_0^\infty \frac{dx}{(x^2+1)^2}.$$

Solution:

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Problem 5B. Complex analysis *Score:*

Show that as the positive integer N tends to infinity, the change in argument of $e^z - z$ is bounded on 3 sides of the square with corners $\pm 2\pi N \pm 2\pi i N$ but is unbounded on the fourth side. Show that $e^z = z$ has infinitely many complex roots.

Solution:

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Problem 6B. Linear algebra *Score:*

Find all eigenvalues and eigenvectors of the linear map $T\colon \mathbb{C}^n \to \mathbb{C}^n$ given by $T((x_1, \ldots, x_n)) = (x_2, x_3, \ldots, x_n, x_1)$.

Solution:

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Problem 7B. Linear algebra *Score:*

Suppose that A and B are linear transformations of a finite dimensional complex vector space such that $AB - BA = A$. If v is an eigenvector of B with eigenvalue λ, show that Av is zero or an eigenvector of B and find its eigenvalue. Prove that A is nilpotent.

Solution:

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Problem 8B. Abstract algebra *Score:*

Let R be a commutative ring with unit. Suppose that there is a monic polynomial $p(x) \in R[x]$ such that the ideal $(p(x)) \subseteq R[x]$ is maximal. Prove that R is a field.

Solution:

STUDENT EXAM NUMBER ________________________

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Problem 9B. Abstract algebra *Score:*

Let M be a (possibly singular) square matrix over a field F. Let p be the product of the non-zero eigenvalues of M (counted with multiplicities) in some algebraically closed extension K of F. Prove that $p \in F$.

Solution: