Problem 1 [20P]

(a) Consider the 3-place truth function \( f : \{T, F\}^3 \to \{0, 1\} \) given by

\[
\begin{array}{ccc|c}
A_0 & A_1 & A_2 & f(A_0, A_1, A_2) \\
T & T & T & T \\
T & T & F & F \\
T & F & T & F \\
T & F & F & T \\
F & T & T & F \\
F & T & F & F \\
F & F & T & F \\
F & F & F & T \\
\end{array}
\]

Find a formula \( \varphi = \varphi(A_0, A_1, A_2) \) such that \( f_\varphi = f \), where \( f_\varphi \) is the truth function derived from \( \varphi \).

You can use the connectives \( \land \) (AND), \( \lor \) (OR), \( \to \), and \( \neg \).

(b) Does there exist a propositional formula \( \varphi \) such that \( \varphi \) is a contradiction and there exists a propositional variable \( A_i \) for which the following holds: If we replace every occurrence of \( A_i \) in \( \varphi \) by \( \neg A_i \), we obtain a new formula \( \varphi' \) that is a tautology?

Give an example or show that such a formula does not exist.

Problem 2 [25P]

(a) Suppose \( \mathcal{M} = (M, I) \) is an \( \mathcal{L_\mathcal{A}} \)-structure. State what it means that a subset \( Y \subseteq M \) is definable from parameters in \( X \subseteq M \).

(b) Let \( \mathcal{A} = \{+, -, \cdot, 0, 1\} \), where \(+, -, \cdot\) are binary function symbols, and \(0, 1\) are constant symbols. Consider the \( \mathcal{L_A} \)-structure \( \mathbb{R} = (\mathbb{R}, +, -, \cdot, 0, 1) \).

- Show that the set \( \{ x \in \mathbb{R} : x > 0 \} \) is definable in \( \mathbb{R} \) (without parameters). Infer that the relation \( \{(x, y) \in \mathbb{R}^2 : x < y\} \) is definable in \( \mathbb{R} \), too.
- Show that for every rational number \( q \), the set \( \{q\} \) is definable without parameters in \( \mathbb{R} \).
- Show that for any two \( a, b \in \mathbb{R}, a \neq b \), the 1-types of \( a \) and \( b \) in \( \mathbb{R} \) are different.

(c – Extra Credit) Give an example of a finite language \( \mathcal{L_A} \) and an infinite \( \mathcal{L_A} \)-structure \( \mathcal{M} \) with exactly four definable (without parameters) subsets of \( M \). Justify your answer.

Problem 3 [15P]

Using the axiom system for first order logic introduced in class (axioms are given below), show that for every term \( \tau \) in \( \mathcal{L} \),

\[ \vdash (\tau \equiv \tau), \]

i.e. there exists a proof of \( (\tau \equiv \tau) \) that uses only the axioms and consequences thereof. Justify your steps.
Problem 4 [40P]

(a) State the *Compactness Theorem* for first order logic.

(b) Suppose \( T_1 \subseteq T_2 \subseteq T_3 \subseteq \ldots \) is a strictly increasing sequence of \( \mathcal{L}_A \)-theories. Suppose further that each \( T_i \) is *closed under logical consequences*, i.e. for each \( i \), if \( T_i \models \sigma \) for some sentence \( \sigma \), then \( \sigma \in T_i \).

- Show that \( \bigcup_{i \in \mathbb{N}} T_i \) is consistent.
- Show that \( \bigcup_{i \in \mathbb{N}} T_i \) is not finitely axiomatizable, i.e. there does not exist a finite set of sentences \( \Gamma \) such that \( M \models \Gamma \) if and only if \( M \models \bigcup_{i \in \mathbb{N}} T_i \).

(c) State the definition of a *complete theory*.

(d) Give an example of a language \( \mathcal{L}_A \) and an \( \mathcal{L}_A \)-theory that is complete.

(e – Extra Credit) Is the theory \( \bigcup_{i \in \mathbb{N}} T_i \) from part (b) necessarily complete? Prove or give a counterexample.

**Axiom System for First Order Logic**

The set of logical axioms, denoted \( \Delta \), is the smallest set of \( \mathcal{L} \) formulas which satisfies the following closure properties.

1. (Instances of Propositional Tautologies) Suppose that \( \phi_1, \phi_2 \) and \( \phi_3 \) are \( \mathcal{L} \) formulas. Then each of the following \( \mathcal{L} \) formulas is a logical axiom:

   (a) \( ((\phi_1 \rightarrow (\phi_2 \rightarrow \phi_3)) \rightarrow ((\phi_1 \rightarrow \phi_2) \rightarrow (\phi_1 \rightarrow \phi_3))) \)

   (b) \( \phi_1 \rightarrow \phi_1 \)

   (c) \( \phi_1 \rightarrow (\phi_2 \rightarrow \phi_1) \)

   (d) \( \phi_1 \rightarrow ((\neg \phi_1) \rightarrow \phi_2)) \)

   (e) \( ((\neg \phi_1) \rightarrow (\phi_1) \rightarrow \phi_1) \)

   (f) \( ((\neg \phi_1) \rightarrow (\phi_1) \rightarrow \phi_2)) \)

   (g) \( (\phi_1 \rightarrow ((\neg \phi_2) \rightarrow (- (\phi_1 \rightarrow \phi_2)))) \)

2. Suppose that \( \phi \) is an \( \mathcal{L} \) formula, \( \tau \) is a term, and that \( \tau \) is substitutable for \( x_i \) in \( \phi \). Then

   \( ((\forall x_i \phi) \rightarrow \phi(x_i; \tau)) \in \Delta. \)

3. Suppose that \( \phi_1 \) and \( \phi_2 \) are \( \mathcal{L} \) formulas. Then

   \( ((\forall x_i (\phi_1 \rightarrow \phi_2)) \rightarrow ((\forall x_i \phi_1) \rightarrow (\forall x_i \phi_2))) \in \Delta. \)

4. Suppose that \( \phi \) is an \( \mathcal{L} \) formula and that \( x_i \) is not a free variable of \( \phi \). Then

   \( (\phi \rightarrow (\forall x_i \phi)) \in \Delta. \)

5. For every variable \( x_i, (x_i = x_i) \in \Delta. \)

6. Suppose that \( \phi_1 \) and \( \phi_2 \) are \( \mathcal{L} \) formulas and that \( x_j \) is substitutable for \( x_i \) in \( \phi_1 \) and in \( \phi_2 \).

   If \( \phi_2(x_i; x_j) = \phi_1(x_i; x_j) \),
   then \( ((x_i = x_i) \rightarrow (\phi_1 \rightarrow \phi_2)) \in \Delta. \)

7. Suppose that \( \phi \in \Delta. \) Then \( (\forall x_i \phi) \in \Delta. \)