Final Exam for MATH 104, Section 1
Fall 2008, December 15, UC Berkeley

Problem 1 [20P]
DEFINITIONS and THEOREMS.
(a) Give three different but equivalent definitions of a continuous function \( f : M \to N \) between metric spaces \( M \) and \( N \).
(b) If \( f_n, f : [a, b] \to \mathbb{R} \), define what it means that \( (f_n) \) converges uniformly to \( f \), \( f_n \Rightarrow f \).
(c) State the Riemann-Darboux Integrability Criterion.
(d) State the Fundamental Theorem of Calculus.

Problem 2 [20P]
EXAMPLES. Give an example of
(a) a compact subset of \( \mathbb{R}^2 \) that is neither homeomorphic to the closed unit disk \( B = \{ x \in \mathbb{R}^2 : \| x \| \leq 1 \} \) nor to the closed interval \([0, 1]\).
(b) a function \( f : \mathbb{R} \to \mathbb{R} \) that is twice differentiable but such that \( f'' \) is not continuous.
(c) a power series \( \sum c_k x^k \) with radius of convergence 2 such that the series converges for \( x = -2 \) but does not converge for \( x = 2 \).
(d) a sequence \( (f_n) \) of continuous functions \( f_n : [0, 1] \to \mathbb{R} \) that does not have a convergent subsequence in \( C^0([0, 1], \mathbb{R}) \) with respect to the sup-metric. (Hence \( C^0([0, 1], \mathbb{R}) \) is not compact.)

You do not have to justify your examples, just state them.

Problem 3 [15P]
Let \((M, d)\) be a metric space.
(a) Show that the union of finitely many compact sets \( K_1, \ldots, K_n \subseteq M \) is compact.
(b) Suppose \( K \subseteq M \) is compact and \( f : K \to \mathbb{R} \) is continuous. Show that \( Z_f = \{ x \in K : f(x) = 0 \} \) is compact.
Problem 4 [15P]

Let \((M, d)\) be a metric space. Given a set \(S \subseteq M\), define the characteristic function \(\chi_S : M \to \{0, 1\}\) of \(S\) as

\[
\chi_S(x) = \begin{cases} 
1 & \text{if } x \in S, \\
0 & \text{if } x \notin S.
\end{cases}
\]

(a) Recall that the boundary of \(S\), \(\partial S\), is defined as \(\partial S = \text{lim}(S) \cap \text{lim}(M \setminus S)\). Show that \(\chi_S\) is discontinuous at \(x\) if and only if \(x \in \partial S\).

(b) Infer that the characteristic function of the Cantor set (as a subset of \([0, 1]\)) is Riemann-integrable.

Problem 5 [15P]

Suppose \(f : \mathbb{R} \to \mathbb{R}\) is continuous.

(a) Show that if \(f\) is differentiable and the derivative \(f'\) is bounded, then \(f\) is uniformly continuous.

(b) Given \(a < b\), argue that \(f\) is Riemann integrable on \([a, b]\) and show that there exists an \(x \in [a, b]\) such that

\[
f(x) = \frac{1}{b-a} \int_a^b f(t) dt.
\]

Problem 6 [15P]

Consider the function \(f : \mathbb{R} \to \mathbb{R}\) defined as

\[
f(x) = \sum_{k=0}^{\infty} \frac{\sin(2kx)}{2^k}.
\]

(a) Argue that this function is well-defined, i.e. that for each \(x \in \mathbb{R}\), \(f(x)\) exists and is finite.

(b) Show that \(f\) is Riemann integrable on any interval \([a, b]\).

(c) Compute

\[
\int_0^\pi f(x) dx.
\]

Extra Credit.

In a metric space \(M\), the interior of a set \(S \subseteq M\), \(\text{int}(S)\), is defined as the set of all points \(s \in S\) for which \(M_r(s) \subseteq S\) for some \(r > 0\). If \(S\) is connected, is \(\text{int}(S)\) connected, too? Prove or give a counterexample.