MATH 115
FINAL EXAM

1 (4 pts)
Solve the simultaneous congruences:

\[ 2x \equiv 5 \pmod{7} \]
\[ 7x \equiv -1 \pmod{11} \]

2 (3 pts) Let \( p \) be a prime, with \( p \equiv 2 \pmod{3} \). Show that, for an integer \( a \) with \( (a, p) = 1 \), the congruence

\[ x^3 \equiv a \pmod{p} \]

always has a unique solution.

3. (8 pts)
Let \( p \) be a prime, and \( a \) be an integer with \( (a, p) = 1 \).

a) (3 pts) Show that \( \{a, 2a, \ldots, (p-1)a\} \) is a reduced residue system for the modulus \( p \).

b) (2 pts) Let \( N_k = 1^k + 2^k + \cdots + (p-1)^k \). Use the results of part a) to show that

\[ a^k N_k \equiv N_k \pmod{p} \]

for any \( a \) with \( (a, p) = 1 \).

Note that the result of part b) can be written as

\[ (a^k - 1)N_k \equiv 0 \pmod{p} \]

for any \( (a, p) = 1 \).

c) (3 pt) Use the result of part b) to show that

\[ N_k \equiv 0 \pmod{p} \]

whenever \( k \) is not divisible by \( p - 1 \).

4. (6 pts) Let \( a \geq 2, k \geq 1 \) be positive integers. Put

\[ n = a^k - 1 \]

It's clear that \( \gcd(a, n) = 1 \).

a) (4 pts) Prove that the order of \( a \pmod{n} \) is equal to \( k \), i.e. \( k \) is the smallest positive integer \( m \), such that

\[ a^m \equiv 1 \pmod{n} \]

(Hint: \( a^m - 1 < a^k - 1 \) for \( m < k \).)

b) (2 pts) Hence show that for \( k \) divides \( \phi(a^k - 1) \), where \( \phi \) is Euler's function.
5. (4 pts) Determine whether
\[ x^2 \equiv 13 \pmod{3019} \]
is solvable, given that 3019 is a prime.

6. (6 pts) List all the positive definite reduced forms of discriminant \(-55\).

7. (6 pts) Compute the quadratic irrationality represented by the periodic continued fraction
\[ \langle 2, 5 \rangle \text{ and } \langle 3, 4 \rangle \]

8. (6 pts) Compute the continued fraction expansion of \(\sqrt{11}\) and \(\sqrt{30}\).

9. (7 pts) Given the continued fraction expansion of \(\sqrt{19}\) is \(\langle 4, 2, 1, 3, 1, 2, 8 \rangle\). Find the smallest positive solution to the equation:
\[ x^2 - 19y^2 = 1 \]